Divisor poset and classical Möbius inversion

$D_n = \{ d \mid d \text{ divides } n \}$, $e \mid d$ if e divides d

Prop. $x \mid y$. Then

$$\mu(x,y) = \begin{cases} 0 & \text{if } y/x \text{ is divisible by square of a prime} \\ (-1)^k & \text{if } y/x \text{ is a product of } k \text{ different primes} \end{cases}$$

Pf. We need to show $\displaystyle\sum_{z \in [x,y]} \mu(x,z) = \delta_{xy}$

Let $x = p_1^{b_1} \cdots p_r^{b_r}$ prime factorizations

$y = p_1^{a_1} \cdots p_r^{a_r}$ $a_i, b_i \geq 0$

$$\sum_{z \in [x,y]} \mu(x,z) = \sum_{\substack{b_1 \leq e_1 \leq a_1 \\ b_2 \leq e_2 \leq a_2 \\ \vdots \\ b_r \leq e_r \leq a_r}} \mu(x, p_1^{e_1} \cdots p_r^{e_r}) = \sum_{\substack{b_1 \leq e_1 \leq \min(b_1+1, a_1) \\ b_2 \leq e_2 \leq \min(b_2+1, a_2) \\ \vdots \\ b_r \leq e_r \leq \min(b_r+1, a_r)}} \mu(x, p_1^{e_1} \cdots p_r^{e_r})$$

If $e_i \geq b_i + 2$, then $\dfrac{p_1^{e_1} \cdots p_r^{e_r}}{x}$ is divisible by p_i^2

Define $\mathbb{P} = \{ p_i \mid b_i < a_i \}$. Then last sum is equal to

$$\sum_{S \subseteq \mathbb{P}} \mu(x, x \cdot \prod_{p \in S} p) = \sum_{S \subseteq \mathbb{P}} (-1)^{|S|} = \sum_{k=0}^{|\mathbb{P}|} (-1)^k \binom{|\mathbb{P}|}{k} = \delta_{0,|\mathbb{P}|}$$

$|\mathbb{P}| = 0 \iff x = y$, so $\delta_{0,|\mathbb{P}|} = \delta_{x,y}$. $\square$

A = alphabet of size k. We consider words of length n up to cyclic shift. Ex. the following are the same:

$$a_1 a_2 a_3 a_4 \equiv a_2 a_3 a_4 a_1 \equiv a_3 a_4 a_1 a_2 \equiv a_4 a_1 a_2 a_3$$

These equivalence classes are called necklaces.

How many necklaces of length n in alphabet of size k?

Note: k^n/n not correct: different shifts could be same:

Ex: $0101 \equiv 1010 \equiv 0101 \equiv 1010$

Period of a word is smallest # of shifts needed to go back to itself. Ex. 0000 period 1

1010 period 2

1234 period 4

Let $\omega(d,k)$ = # words of period d in alphabet of size k

$$\#\text{ necklaces of length } 4 = \omega(1,k) + \frac{\omega(2,k)}{2} + \frac{\omega(4,k)}{4}$$

$$\#\text{ necklaces of length } n = \sum_{d|n} \frac{\omega(d,k)}{d}$$

Thm (Witt's formula) For any $d>0$,

$$\omega(d,k) = \sum_{e|d} \mu(e,d)\, k^e.$$

Pf. Let $g(e)$ = # words of length d whose period divides e

$f(e)$ = # ______ exactly e

$g(e) = k^e$ since such words determined by first e letters and no other conditions

Also $g(e) = \sum_{e'|e} f(e') \xrightarrow[\text{inversion}]{\text{Möbius}} f(d) = \sum_{e|d} \mu(e,d)\, k^e$ $\square$

Ex. $n=4$: $\omega(1,k) = \mu(1,1)\, k = k$

$\omega(2,k) = \mu(1,2)\, k + \mu(2,2)\, k^2 = -k + k^2$

$\omega(4,k) = \mu(1,4)\, k + \mu(2,4)\, k^2 + \mu(4,4)\, k^4 = -k^2 + k^4$

$\Rightarrow$ # necklaces of length 4 $= \omega(1,k) + \dfrac{\omega(2,k)}{2} + \dfrac{\omega(4,k)}{4}$

$$= k + \frac{k^2-k}{2} + \frac{k^4-k^2}{4} = \frac{k^4+k^2+2k}{4}$$

$n=6$ $\quad \omega(3,k) = \mu(1,3)k + \mu(3,3)k^3 = -k+k^3$

$$\omega(6,k) = \mu(1,6)k + \mu(2,6)k^2 + \mu(3,6)k^3 + \mu(6,6)k^6$$
$$= k - k^2 - k^3 + k^6$$

necklaces of length 6 $= \omega(1,k) + \dfrac{\omega(2,k)}{2} + \dfrac{\omega(3,k)}{3} + \dfrac{\omega(6,k)}{6}$

$$= k + \frac{k^2-k}{2} + \frac{k^3-k}{3} + \frac{k^6-k^3-k^2+k}{6}$$

$$= \frac{k^6+k^3+2k^2+2k}{6}$$

EX. $i = \sqrt{-1}$. Consider polynomial $x^d - 1$

let $\omega_d = e^{2\pi i/d}$. By Euler's formula, $e^{2\pi i} = 1$.

So $1, \omega_d, \omega_d^2, \dots, \omega_d^{d-1}$ are the distinct roots of x^d-1

If d' divides d, then $x^{d'}-1$ divides x^d-1

$\Phi_d(x) =$ dth cyclotomic polynomial

$=$ result of dividing x^d-1 by $(x-\omega_d^j)$ where $\gcd(j,d) \neq 1$.

Then $\Phi_d(x)$ and $x^{d'}-1$ have no common factors when $d' < d$ and $d' | d$

$$\Rightarrow \quad x^d - 1 = \prod_{d'|d} \Phi_{d'}(x)$$

$\xrightarrow{\text{Möbius inversion}}$ $\quad \Phi_d(x) = \prod_{d'|d} (x^{d'}-1)^{\mu(d',d)}$

$\underline{d=6:}$

$$(x-\omega_6)(x-\omega_6^5) = \Phi_6(x) = \frac{(x^6-1)(x-1)}{(x^3-1)(x^2-1)} = x^2-x+1$$