Möbius Inversion

Def. A partially ordered set (poset) P is a set w/ a relation, denoted $\leq$, s.t.

1. (Reflexive) $x \leq x \quad \forall x \in P$
2. (Transitive) $x \leq y$ & $y \leq z \Rightarrow x \leq z \quad \forall x, y, z \in P$
3. (Antisymmetric) $x \leq y$ & $y \leq x \Rightarrow x = y \quad \forall x, y \in P$

$x < y$ means $x \leq y$ & $x \neq y$

Ex. ① $P = [n]$, $x \leq y$ has usual meaning, total ordering

② S = set, P = all subsets of S, $x \leq y$ if x is a subset of y — Boolean poset

not a total ordering if $|S| \geq 2$

③ $P = \mathbb{Z}_{>0}$ $x \leq y$ means x divides y (use $x | y$)

For $n \in \mathbb{Z}_{>0}$, let $D_n = \{x \mid x \text{ divides } n\}$

divisor poset

④ P = all set partitions of $[n]$

X refines y if every block of x is a subset of some block of y

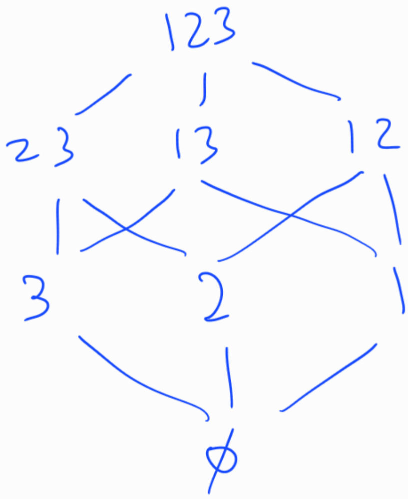
$x \leq y$ means x refines y

ex: $12|34|5$ refines $125|34$, but not $134|25$

Hasse diagram of poset

Draw elements as vertices, draw line $\begin{matrix} x \\ | \\ y \end{matrix}$ to mean $x \geq y$ & this is a cover (no other z exist s.t. $x \geq z$ & $z \geq y$)

Ex. Boolean poset on $\{1,2,3\}$



Def. $[x,y] = \{z \mid x \le z \;\&\; z \le y\}$

$(x,y) = \{z \mid x < z \;\&\; z < y\}$

$[x,y) = \{z \mid x \le z \;\&\; z < y\}$

$(x,y] = \{z \mid x < z \;\&\; z \le y\}$

For $x \le y$ define $\mu(x,y)$ as follows: ($\underline{\text{Möbius function}}$) (assume P finite)

$$\mu(x,x) = 1 \quad \forall x \in P$$

$$\mu(x,y) = -\sum_{z \in [x,y)} \mu(x,z) \quad \text{if } x<y$$

Note: This implies $\displaystyle\sum_{z \in [x,y]} \mu(x,z) = \delta_{xy} = \begin{cases} 1 & \text{if } x=y \\ 0 & \text{if } x \ne y \end{cases}$

(this also characterizes μ)

Lemma. $$\sum_{z \in [x,y]} \mu(z,y) = \delta_{xy}$$

Pf. Define μ' as follows:

$$\mu'(x,x) = 1 \quad \forall x \in P$$

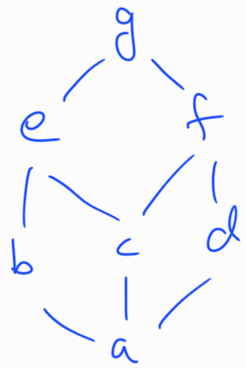
$$\mu'(x,y) = -\sum_{z \in (x,y]} \mu'(z,y) \quad \text{for } x<y$$

$$\mu(x,y) = \sum_{w\in[x,y]} \mu(x,w)\,\delta_{wy} = \sum_{w\in[x,y]} \mu(x,w) \sum_{z\in[w,y]} \mu'(z,y)$$

$$= \sum_{z\in[x,y]} \mu'(z,y) \sum_{w\in[x,z]} \mu(x,w) = \sum_{z\in[x,y]} \mu'(z,y)\,\delta_{xz} = \mu'(x,y).$$

$\Rightarrow$ $\mu = \mu'$, so μ satisfies the relation in question. $\square$

Ex. P:



$\mu(a,b) = -\mu(a,a) = -1$

$\mu(a,c) = \mu(a,d) = -1$

$\mu(a,e) = -\big(\mu(a,a) + \mu(a,b) + \mu(a,c)\big) = -(1-1-1) = 1$

$\mu(a,f) = 1$ by symmetry

$\mu(a,g) = -\big(\mu(a,a) + \mu(a,b) + \mu(a,c) + \mu(a,d) + \mu(a,e) + \mu(a,f)\big)$

$= -(1 - 1 - 1 - 1 + 1 + 1) = 0$

Thm (Möbius inversion formula). Let P = poset (finite)

f, g functions on P

(a) $g(y) = \sum_{x\le y} f(x)$ $\forall y\in P$ $\Longleftrightarrow$ $f(y) = \sum_{x\le y} g(x)\mu(x,y)$ $\forall y\in P$

(b) $g(y) = \sum_{x\ge y} f(x)$ $\forall y\in P$ $\Longleftrightarrow$ $f(y) = \sum_{x\ge y} g(x)\mu(y,x)$ $\forall y\in P$

pf. (a) Suppose $g(y) = \sum_{x \leq y} f(x)$. $\forall y \in P$. Pick y. Then

$$\sum_{x \leq y} g(x)\mu(x,y) = \sum_{x \leq y} \mu(x,y) \sum_{z \leq x} f(z) = \sum_{z \leq y} f(z) \sum_{x \in [z,y]} \mu(x,y)$$

$$= \sum_{z \leq y} f(z)\, \delta_{zy} = f(y)$$

Now suppose $f(y) = \sum_{x \leq y} g(x)\mu(x,y)$ $\forall y \in P$

$$\sum_{x \leq y} f(x) = \sum_{x \leq y} \sum_{z \leq x} g(z)\mu(z,x) = \sum_{z \leq y} g(z) \sum_{x \in [z,y]} \mu(z,x)$$

$$= \sum_{z \leq y} g(z)\, \delta_{zy} = g(y).$$

$\square$

(b) is similar.