Cayley's Enumeration of Labeled Trees

finite set S, labeled (simple) graph = graph whose vertex set is S

cycle = closed walk that does not repeat edges

forest = graph w/ no cycles

tree = connected forest, t_n = # labeled trees w/ n vertices

Thm (Cayley) $t_n = n^{n-2}$

Ex. $n=1$ or $n=2$, $t_1 = t_2 = 1$

1   1 — 2

$n=3$, $t_3 = 3$ 1 — 2 — 3   2 — 1 — 3   1 — 3 — 2

$n=4$, $t_4 = 16$

4

$\frac{4!}{2} = 12$

Rooted labeled tree = labeled tree w/ distinguished choice of vertex (the root)

rooted labeled trees w/ n vertices $= n t_n$

Planted labeled forest = labeled forest w/ distinguished choice of vertex in each connected component
= disjoint union of rooted labeled trees

f_n = # planted labeled forests w/ n vertices.

$$F(x) = \sum_{n\geq 0} f_n \frac{x^n}{n!}, \quad R(x) = \sum_{n\geq 0} n t_n \frac{x^n}{n!}$$

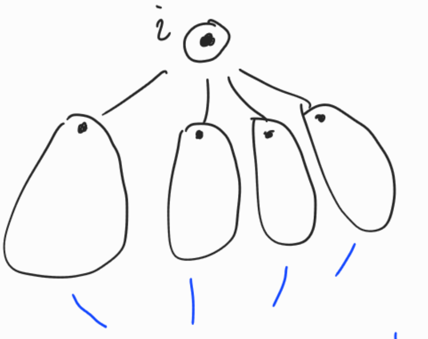
$f_0 = 1$

Lemma $F(x) = e^{R(x)}$

Pf. Every planted labeled forest is disjoint union of rooted labeled trees. $\square$

Lemma. $R(x) = xF(x)$.

Pf. For $n \geq 1$, every rooted labeled tree on $[n]$ can be obtained uniquely as follows:



Pick $i \in [n]$ to be root. Then we put structure of planted labeled forest on $[n] \setminus i$. Then connect i to the roots of each component.

Conversely, given a rooted labeled tree, let i be its root. Then $[n] \setminus i$ is a labeled forest (in fact planted).

$$\Rightarrow [x^n]\, n!\, R(x) = n t_n = n f_{n-1} = n!\, [x^{n-1}] F(x) = n!\, [x^n]\, x F(x)$$

$$\Rightarrow \quad R(x) = xF(x). \quad \square$$

$$\Rightarrow \quad R(x) = x e^{R(x)}. \qquad \text{let } r_n = \frac{n t_n}{n!} = [x^n] R(x)$$

$$R(x) = x\left(1 + R(x) + \frac{R(x)^2}{2!} + \frac{R(x)^3}{3!} + \cdots\right)$$

$$r_0 = 0 \Rightarrow [x^n] R(x) = [x^{n-1}]\left(1 + R(x) + \cdots + \frac{R(x)^{n-1}}{(n-1)!}\right)$$

r_n is uniquely determined by $r_0, r_1, \ldots r_{n-1}$

$$r_1 = [x^1]R(x) = [x^0]\,1 = 1$$
$$r_2 = [x^2]R(x) = [x^1](1+R(x)) = r_1 = 1$$
$$r_3 = [x^3]R(x) = [x^2]\left(1+R(x)+\frac{R(x)^2}{2}\right) = 0 + r_2 + \frac{r_1^2}{2} = 1+\frac{1}{2} = \frac{3}{2}$$
$$\vdots$$

<u>Thm</u> (Lagrange inversion formula). Let $G(x)$ be formal power series s.t. $G(0) \neq 0$. Let $A(x)$ formal power series s.t. $A(0)=0$, and

$$A(x) = x\,G(A(x)).$$

Then for k, n we have

$$n[x^n]A(x)^k = k[x^{n-k}](G(x)^n).$$

<u>Pf of Cayley</u>: Apply Lagrange w/ $A(x) = R(x)$, $G(x) = e^x$.

$$n[x^n]R(x)^k = k[x^{n-k}]e^{nx} = k[x^{n-k}]\sum_{d\geq 0}\frac{(nx)^d}{d!}$$
$$= k\,\frac{n^{n-k}}{(n-k)!}$$

If $n>0$, divide by n: $[x^n]R(x)^k = \frac{k}{n}\,\frac{n^{n-k}}{(n-k)!}$

Let $k=1$, get $\frac{n t_n}{n!} = [x^n]R(x) = \frac{1}{n}\,\frac{n^{n-1}}{(n-1)!}$

$$\Rightarrow t_n = n^{n-2}$$ $\square$

<u>Cor</u>. # planted labeled forests on n vertices w/ k connected components is $\binom{n-1}{k-1}n^{n-k}$

<u>Pf</u>. $n![x^n]R(x)^k =$ # planted labeled forests on n vertices w/ k connected components + ordering of components

$$\frac{n!}{k!}[x^n]\,R(x)^k = \frac{n!}{k!}\,\frac{k}{n}\,\frac{n^{n-k}}{(n-k)!} = \frac{(n-1)!}{(k-1)!\,(n-k)!}\,n^{n-k} = \binom{n-1}{k-1}n^{n-k}. \square$$