12-fold way, summary

k balls / n boxes	f arbitrary	f injective	f surjective
dist/dist	words n^k	injective words $(n)_k$	ordered set partitions $n!\,S(k,n)$
indist/dist	weak compositions $\binom{n+k-1}{k}$	subsets $\binom{n}{k}$	compositions $\binom{k-1}{n-1}$
dist/indist	Set partitions of [K] w/ $\le n$ blocks $\sum_{i=1}^{n} S(k,i)$	$\times$	Set partitions of [K] w/ n blocks $S(k,n)$
indist/indist	integer partitions of k w/ $\le n$ parts $P_{\le n}(k)$	$\times$	integer partitions of k w/ n parts $P_n(k)$

Cycles in permutations

$(x)_k = x(x-1)(x-2)\cdots(x-k+1)$

Prop. $$x^n = \sum_{k=0}^{n} S(n,k)\,(x)_k$$

Pf. Pick positive integer $d \ge n$.

$d^n = \#$ functions $[n] \to [d]$

Count such functions by their image $S \subseteq [d]$

$= \#$ surjective functions $[n] \to S = |S|!\, S(n,|S|)$

$$\Rightarrow d^n = \sum_{S \subseteq [d]} |S|!\, S(n,|S|) = \sum_{k=0}^{d} k!\, S(n,k)\binom{d}{k} = \sum_{k=0}^{d} S(n,k)\,(d)_k$$

$$\sum_{k=0}^{n} S(n,k)\,(d)_k$$

If $k > n$, then $S(n,k) = 0$

$\Rightarrow x^n - \sum_{k=0}^{n} S(n,k)(x)_k$ has infinitely many roots

Nonzero polynomials only have finitely many roots

$\Rightarrow \quad x^n = \sum_{k=0}^{n} S(n,k)\,(x)_k$ $\square$

How to write $(x)_k$ as sum of x^n?

Cycle decomposition of permutation $\sigma \in \mathfrak{S}_n$:

given $1 \le i \le n$, consider sequence $i, \sigma(i), \sigma^2(i), \ldots \sigma^{k-1}(i)$

where $\sigma^k(i) = i$ Denote $i \to \sigma(i) \to \sigma^2(i) \to \cdots \to \sigma^{k-1}(i) \to i$

Length 1 cycles are possible.

Ex. $\sigma = 135624$

$1 \to 1$
$2 \to 3 \to 5 \to 2$
$4 \to 6 \to 4$

3 cycles

$c(n,k) = \#$ permutations $\sigma \in \mathfrak{S}_n$ w/ k cycles

(signless) Stirling numbers of the first kind

$c(0,0) = 1$.

Note: $c(n,0) = 0$ if $n > 0$

Prop. $n \ge k \ge 1$: $c(n,k) = c(n-1,k-1) + (n-1) \cdot c(n-1,k)$

Pf. Consider 2 types of permutations w/ k cycles.

Type I. Permutations where n is its own cycle.

Delete n, get permutation in $\mathfrak{S}_{n-1}$ w/ $k-1$ cycles.

Can recover permutation uniquely, so get bijection.

There are $c(n-1,k-1)$ many permutations.

Type II. Permutations where n is not in its own cycle.

$\cdots \; i \to n \to j \to \cdots$ $\quad i \neq n$, $j \neq n$. Let σ be such a permutation.

Define $\tau \in \mathfrak{S}_{n-1}$ by $\tau(i) = j$

$\tau(x) = \sigma(x)$ for all $x \neq i$.

τ has k cycles, σ can be recovered if I remember i.

Get bijection $\left\{ \begin{array}{l} \text{type II} \\ \text{permutations} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{l} \text{pairs } (i, \tau) \\ 1 \leq i \leq n-1,\ \tau \in \mathfrak{S}_{n-1} \\ \text{has } k \text{ cycles} \end{array} \right\}$

$\#$ type II $= (n-1)\, c(n-1,k)$

$$\Longrightarrow \quad c(n,k) = c(n-1,k-1) + (n-1)\, c(n-1,k). \qquad \square$$

Cor. For $n \geq 0$, $\displaystyle\sum_{k=0}^{n} c(n,k)\, x^k = x(x+1)\cdots(x+n-1)$.

where right side is 1 if $n=0$. In particular,

$$\sum_{k=0}^{n} (-1)^{n-k} c(n,k)\, x^k = (x)_n.$$

Pf. Prove by induction on n. If $n=0$, get $1=1$ ✓

Now suppose $n>0$. Then $c(n,0)=0$. We get

$$\sum_{k=1}^{n} c(n,k)\, x^k = x \sum_{k=1}^{n} c(n-1,k-1)\, x^{k-1} + (n-1) \sum_{k=1}^{n} c(n-1,k)\, x^k$$

$j = k-1$

$$= x \sum_{j=1}^{n-1} c(n-1,j)\, x^j + (n-1) \sum_{j=1}^{n-1} c(n-1,j)\, x^j$$

$$= (x+n-1) \sum_{j=1}^{n-1} c(n-1,j)\, x^j = (x+n-1) \cdot x(x+1)\cdots(x+n-2)$$

✓

Substitute $x \to -x$

$$\sum_{k=0}^{n} (-1)^k c(n,k)\, x^k = (-x)(-x+1)\cdots(-x+n-1)$$

Multiply by $(-1)^n$:

$$\sum_{k=0}^{n} (-1)^{n-k} c(n,k)\, x^k = x(x-1)\cdots(x-n+1) = (x)_n. \qquad \square$$

<u>Def.</u> $s(n,k) = (-1)^{n-k} c(n,k)$ Stirling numbers of first kind.

<u>Cor</u> ("inversion formula") For $n, \ell \geq 0$, we have:

$$\sum_{k=0}^{n} S(n,k)\, s(k,\ell) = \delta_{n,\ell} = \sum_{k=0}^{n} s(n,k)\, S(k,\ell)$$

<u>Pf.</u> $x^n = \sum_{k=0}^{n} S(n,k)(x)_k = \sum_{k=0}^{n} S(n,k) \sum_{\ell=0}^{k} s(k,\ell)\, x^\ell$

Take coefficient of x^ℓ:

Left side: $\delta_{n,\ell}$

right side: $\sum_{k=0}^{n} S(n,k)\, s(k,\ell)$

<u>Second identity:</u> $(x)_n = \sum_{k=0}^{n} s(n,k)\, x^k = \sum_{k=0}^{n} s(n,k) \sum_{\ell=0}^{k} S(k,\ell)\, (x)_\ell$

$(x)_n$ form basis for polynomials as n varies.

Ask for coefficient of $(x)_\ell$ of both sides

left side: $\delta_{n\ell}$, right side: $\sum_{k=0}^{n} s(n,k)\, S(k,\ell)$. $\square$