Words

Def. A <u>word</u> is a finite sequence w/ entries in a set A (= <u>alphabet</u>). The <u>length</u> of word is length of sequence.

There is a unique empty word of length 0.

<u>12-fold:</u> A = set of distinguishable boxes
k = number of distinguishable balls.

A word in A of length $k \longleftrightarrow$ assignment of balls to boxes
entry i records which box the ith ball goes to

<u>Ex.</u> $A = \{a, b\}$. The words of length ≤ 2:

$\emptyset$, a, b, aa, ab, ba, bb

<u>Thm.</u> If $|A| = n$, there are n^k words of length k.

<u>Pf.</u> words of length k are elements in $A^k = \underbrace{A \times \cdots \times A}_{k}$

$|A^k| = |A|^k = n^k$.

Alternatively, we can build words by picking entries one by one, each entry has n choices, can be chosen independently $\rightsquigarrow n^k$. $\square$

Generating functions:

- Fix $n = |A|$, vary k: $\displaystyle\sum_{k \geq 0} \#\text{words in } A \text{ of length } k \; x^k = \sum_{k \geq 0} n^k x^k = \frac{1}{1 - nx}$.

- Fix k, vary $n = |A|$: $\displaystyle\sum_{n \geq 0} n^k x^n = \frac{A_k(x)}{(1-x)^{k+1}} \longleftarrow$ polynomial of degree $\leq k$

Eulerian polynomials

Ex. A city w/ 10 intersections

Each one may have: nothing, stop light, gas station, or both

Every possible configuration is encoded by word of length 10 in an alphabet of size 4. $\rightsquigarrow$ 4^{10} possibilities.

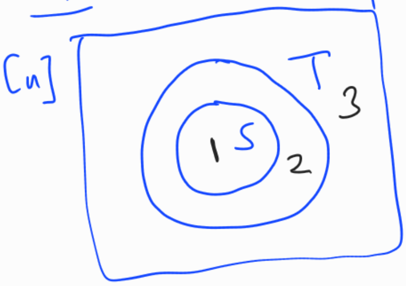
Ex. Counting subsets. $A = \{0,1\}$

$$\left\{ \begin{matrix} \text{subsets of} \\ [n] \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{words of length} \\ n \text{ in } A \end{matrix} \right\}$$

$$S \subseteq [n] \longrightarrow w_S = i\text{th entry is } \begin{matrix} 0 \text{ if } i \notin S \\ 1 \text{ if } i \in S \end{matrix}$$

$\rightsquigarrow$ another derivation that #subsets is 2^n.

Ex. Count pairs of subsets $S, T \subseteq [n]$ s.t. $S \subseteq T$.



① in S (and also T)

② in T but not S

③ not in T (and also not S)

$A = \{①, ②, ③\}$

$$\left\{ \begin{matrix} \text{pairs of subsets} \\ S \subseteq T \subseteq [n] \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{words of length} \\ n \text{ in } A \end{matrix} \right\}$$

$\Longrightarrow$ # pairs is 3^n.

Another approach:

$$\#\text{pairs} = \sum_{T \subseteq [n]} \sum_{S \subseteq T} 1 = \sum_{T \subseteq [n]} \begin{matrix}\#\text{subsets} \\ \text{in } T\end{matrix} = \sum_{T \subseteq [n]} 2^{|T|} = \sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n.$$

Injective words: words w/ no repetitions in entries.

12-fold: asking for assignment of balls into boxes to be injective.

Given $n \geq k$, define falling factorial [when $n = k$, $(n)_n = n!$]

$$(n)_k := n(n-1)(n-2)\cdots(n-k+1)$$

Thm. # of injective words of length k in an alphabet of size n is $(n)_k$ if $n \geq k$, and 0 else.

Pf. Assume $n \geq k$. Pick a permutation of A.

The first k entries is an injective word.

Redundancy: each injective word appears $(n-k)!$ times

$$\# \text{ injective words} = \frac{n!}{(n-k)!} = (n)_k \qquad \square$$

Generating functions: Fix k, sum over n:

$$\sum_{n \geq k} (n)_k x^n = k! \sum_{n \geq k} \binom{n}{k} x^n$$

Note: $$\frac{k!\, x^k}{(1-x)^{k+1}} = k! \sum_{m \geq 0} \binom{m+k}{k} x^{m+k} = k! \sum_{n \geq k} \binom{n}{k} x^n$$

$n = m+k$

Note: $\{n^k \mid k \geq 0\}$ is "natural" basis for polynomials in n
but generating fcn very complicated

$\{(n)_k \mid k \geq 0\}$ is also basis for polynomials in n
"less natural" but generating functions are simpler.