Math 188, Spring 2021
Homework 8
Due: May 28, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) How many ways are there to list the letters of the word MATTDAMON so that no two consecutive letters are the same?

(2) Let $n > 1$ be an integer. We have n married couples ($2n$ people in total).
   (a) How many ways can we have the $2n$ people stand in a line so that no person is standing next to their spouse?
   (b) Same as (a), but replace line by circle.

(3) Let q be a prime power and n a positive integer. Let V be an n-dimensional $\mathbf{F}_q$-vector space and let P be the poset whose elements are linear subspaces of V with the ordering $X \le Y$ if X is contained in Y. Show that the Möbius function of P is given by

$$\mu(X, Y) = (-1)^d q^{\binom{d}{2}}$$

where $d = \dim Y - \dim X$. Hint at end.

(4) For a positive integer n, define

$$f(n) = |\{i \in \mathbf{Z} \mid 1 \le i \le n,\ \gcd(n, i) = 1\}|.$$

   (a) Show that

$$n = \sum_{d|n} f(d)$$

   where the sum is over all positive integers d that divide n.
   (b) Use Möbius inversion to show that

$$f(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

   where the product is over the primes p that divide n.

(5) Let Π_n be the poset of set partitions of $[n]$ and let μ be its Möbius function. Write a formula for the number of connected labeled graphs with vertex set $[n]$ using μ (a formula for μ is given in the book, but you don't need to look it up).

1. Optional problems (don't turn in)

(6) A derangement of a set S is a bijection $f \colon S \to S$ such that $f(i) \ne i$ for $i \in S$. Let d_n be the number of derangements of $[n]$, and let

$$D(x) = \sum_{n \ge 0} \frac{d_n}{n!} x^n.$$

   (a) Using the structure interpretation for products of EGF, show that

$$D(x)e^x = \frac{1}{1-x}.$$

(b) Show how this implies the formula we previously obtained:

$$d_n = \sum_{i=0}^{n} (-1)^i \frac{n!}{i!}.$$

(7) There are n people sitting at a circular table. How many ways can they rearrange seats so that no one sits next to someone they were sitting next to before?

(8) Let q be a prime power and let N_n be the number of monic irreducible polynomials of degree n with coefficients in $\mathbf{F}_q$.

(a) Using that polynomials over a field satisfy unique factorization, show that

$$(1 - qx)^{-1} = \prod_{d \geq 1} (1 - x^d)^{-N_d}.$$

(b) Take the logarithmic derivative of (a) and compare the coefficient of x^{n-1} to get $q^n = \sum_{d|n} dN_d$.

(c) Use Möbius inversion to get a formula for N_n.

2. Hints

(3) For subspaces $X \subseteq Y$, the set of r-dimensional subspaces Z such that $X \subseteq Z \subseteq Y$ are in bijection with $(r - \dim X)$-dimensional subspaces in the quotient space Y/X, and $\dim(Y/X) = \dim Y - \dim X$.