Math 188, Spring 2021
Homework 4
Due: April 30, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

Reminder: First part of project also due April 30. Last minute submissions run the risk of conflicting, in which case no credit is given.

`http://www.math.ucsd.edu/~ssam/188/project.html`

(1) Let n be a positive integer. Consider the equation

$$x_1 + x_2 + \cdots + x_8 = 2n.$$

For each of the following conditions, how many solutions are there? Give as simple of a formula as possible. (Each part is an independent problem, don't combine the conditions.)

(a) The x_i are non-negative even integers.

(b) The x_i are positive odd integers.

(c) The x_i are non-negative integers and $x_8 \le 10$.

(2) Let k, n be positive integers.

(a) Show that

$$\sum_{(a_1,\dots,a_n)} a_1 a_2 \cdots a_n = \binom{n+k-1}{k-n}$$

where the sum is over all compositions of k into n parts. (Hint at end.)

(b) Show that

$$\sum_{(a_1,\dots,a_n)} 2^{a_2-1} 3^{a_3-1} \cdots n^{a_n-1} = S(k,n)$$

where the sum is over all compositions of k into n parts.

(3) (a) Give a closed formula for the number of pairs of subsets S, T of $[n]$ such that $S \subsetneq T$ (i.e., $S \subset T$ and $S \neq T$).

(b) Give a closed formula for the number of k-tuples of subsets $(S_1, \dots, S_k)$ of $[n]$ such that $\bigcap_{i=1}^k S_i = \emptyset$.

(4) (a) Let r be a fixed positive integer. Show that $S(n+r, n)$ is a polynomial function of n of degree $2r$ for $n \ge 0$.

(b) Compute this polynomial for $r = 2, 3, 4$.

(5) Let $F(n)$ be the number of set partitions of $[n]$ such that every block has size ≥ 2. Prove that

$$B(n) = F(n) + F(n+1),$$

where $B(n)$ is the nth Bell number.

1. Optional problems (don't turn in)

(6) What is the total number of parts of all compositions of k?

[For example, when $k = 2$, the only compositions are (2) and $(1, 1)$ so there are a total of 3 parts.]

(7) Fix an integer $k \ge 2$. Call a composition $(a_1, \dots, a_n)$ of k **doubly even** if the number of a_i which are even is also even (i.e., there could be no even a_i, or 2 of them, or 4, etc.). Show that the number of doubly even compositions of k is 2^{k-2}.

For example, if $k = 4$, then here are the 4 doubly even compositions of 4:

$$(2, 2), \qquad (3, 1), \qquad (1, 3), \qquad (1, 1, 1, 1).$$

(8) Give a closed formula for the number of k-tuples of subsets $(S_1, \ldots, S_k)$ of $[n]$ such that $S_i \subseteq S_{i+1}$ for $i = 1, \ldots, k-1$.

2. Hints

(2)a: Consider the product

$$\left(\sum_{a_1 \geq 1} a_1 x^{a_1}\right) \cdots \left(\sum_{a_n \geq 1} a_n x^{a_n}\right).$$

(7): Given a composition $\alpha = (a_1, \ldots, a_n)$, define another composition $\Phi(\alpha)$ by

$$\Phi(\alpha) = \begin{cases} (1, a_1 - 1, a_2, a_3, \ldots, a_n) & \text{if } a_1 > 1 \\ (a_2 + 1, a_3, \ldots, a_n) & \text{if } a_1 = 1 \end{cases}.$$

(in both cases, we didn't do anything to $a_3, \ldots, a_n$). Show that Φ defines a bijection between the set of doubly even compositions of k and the set of compositions of k which are not doubly even.