Math 188, Spring 2021
Homework 1
Due: April 9, 2021 11:59PM via Gradescope

Solutions must be **clearly** presented. Incoherent or unclear solutions will lose points.

(1) Find a closed formula for the following recurrence relation:

$$\begin{aligned} a_0 &= 1,\ a_1 = 1,\ a_2 = 2, \\ a_n &= 5a_{n-1} - 8a_{n-2} + 4a_{n-3} \qquad (n \geq 3). \end{aligned}$$

(2) Find a closed formula for the following recurrence relation:

$$\begin{aligned} a_0 &= 1, \\ a_n &= 3a_{n-1} + 2^n \qquad (n \geq 1). \end{aligned}$$

(3) Let $r_1, \dots, r_d$ be distinct numbers. Prove that the sequences $(r_1^n), \dots, (r_d^n)$ are linearly independent by showing that the determinant of $(r_i^{j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$).

(4) Let $(a_n)_{n\geq 0}$ be a sequence satisfying a linear recurrence relation whose characteristic polynomial is $(t^2-1)^d$.
   (a) Show that there exist polynomials $p(n)$ and $q(n)$ of degree $\leq d-1$ such that
$$a_n = \begin{cases} p(n) & \text{if } n \text{ is even} \\ q(n) & \text{if } n \text{ is odd} \end{cases}.$$
   (b) How does this generalize if the characteristic polynomial is $(t^k-1)^d$?

(5) (a) Suppose that $(a_n)_{n\geq 0}$ and $(a'_n)_{n\geq 0}$ both satisfy the same linear recurrence relation of order d and that they agree in d consecutive places, i.e., there exists k such that $a_k = a'_k, a_{k+1} = a'_{k+1}, \dots, a_{k+d-1} = a'_{k+d-1}$. Show that these sequences are the same.
   (b) Suppose that $(a_n)_{n\geq 0}$ satisfies the linear recurrence relation of order d
$$a_n = c_1 a_{n-1} + \cdots + c_d a_{n-d} \qquad \text{for all } n \geq d.$$
   Show that there is a unique sequence $(b_n)_{n\in\mathbf{Z}}$ (indexed by *all* integers) such that $b_n = a_n$ for $n \geq 0$ and such that
$$b_n = c_1 b_{n-1} + \cdots + c_d b_{n-d} \qquad \text{for all } n \in \mathbf{Z}.$$
   Explain how to get a closed form formula for b_n.
   (c) Consider the Fibonacci sequence $f_0 = 0$, $f_1 = 1$, and $f_{n+2} = f_{n+1} + f_n$. How does the negatively indexed Fibonacci sequence relate to the usual one?

1. Optional problems (don't turn in)

(1) Let p be a prime number and let $(a_n)_{n\geq 0}$ be a sequence such that $a_n \in \mathbf{Z}/p$ and which satisfies a homogeneous linear recurrence relation. Show that the sequence is in fact periodic.

(2) Let $r_1, \dots, r_{d-1}$ be distinct numbers. Prove that the sequences $\alpha_1 = (r_1^n), \dots, \alpha_{d-1} = (r_{d-1}^n), \alpha_d = (nr_{d-1}^{n-1})$ are linearly independent by showing that the determinant of $(\alpha_{i,j-1})_{i,j=1,\dots,d}$ is nonzero (interpret $0^0 = 1$ and if $r_{d-1} = 0$, interpret $\alpha_{d,0} = 0$).