NOTES FOR MATH 251C

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The topic of the course is representations of Lie groups, with an emphasis on classical (general linear, symplectic, orthogonal) groups. We will explore this from several different angles: linear algebra, invariant theory, combinatorics, algebraic geometry. The perspective I will take is that the representation theory of general linear groups is a natural extension of linear algebra and for symplectic and orthogonal groups, we are doing linear algebra in the presence of a (skew-)symmetric bilinear form. I intend to take advantage of theorems from the general theory (I will recall the necessary background as we go), but our focus will be to explore examples and explicit constructions rather than the proofs of such theorems.

I'm not using any particular reference, but [FH] comes the closest to what I want to cover.

For simplicity, we'll work with groups over the complex numbers. Some of what we say will work over different fields. Another disclaimer: I don't intend for this to be a comprehensive reference. So results will usually be stated in special cases the first time even if a more general case is needed later. Instead, I will try to minimize technical definitions for later cases until they are needed, and then restate results in more general forms.

Contents

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1. General linear group

Let V be a finite-dimensional complex vector space. We let $\mathbf{GL}(V)$ denote the group of invertible linear operators $V \to V$ where the group operation is composition. If $V = \mathbf{C}^n$, then we write $\mathbf{GL}_n(\mathbf{C})$ instead of $\mathbf{GL}(V)$. The only difference is that $\mathbf{C}^n$ has a chosen basis (the standard one $e_1, \dots, e_n$) while V does not. This is a Lie group (in fact, a complex Lie group).

1.1. Representations.

Some basic notation we will use over and over again with regards to $\mathbf{GL}_n(\mathbf{C})$: we let $B \subset \mathbf{GL}_n(\mathbf{C})$ be the subgroup of upper-triangular matrices (called a **Borel subgroup**) and $T \subset \mathbf{GL}_n(\mathbf{C})$ be the subgroup of diagonal matrices (called a **maximal torus**). When talking about $\mathbf{GL}(V)$, a subgroup B is a Borel subgroup if it is the group of upper-triangular matrices with respect to some choice of basis for V. Similarly, T is a maximal torus if it is the group of diagonal matrices with respect to some choice of basis for V. So note that T and B are not unique, but choosing a basis will determine a choice (and this will be convenient when we want to do calculations).

An **algebraic representation** (or **rational representation**) of $\mathbf{GL}(V)$ is a group homomorphism $\rho\colon \mathbf{GL}(V) \to \mathbf{GL}(W)$ for some other finite-dimensional complex vector space W which is algebraic: this means that for some (equivalently, any) choice of ordered bases for V and W, then for any $g \in \mathbf{GL}(V)$ the entries of $\rho(g)$ are all rational functions of the entries of g. All representations that we consider will be algebraic, so we will just say representation. If the entries are all polynomial functions, then we call ρ a **polynomial representation**.

Example 1.1.1.

- If $V = W$ we can take ρ to be the identity.
- If $V = W = \mathbf{C}^n$, take $\rho(g) = (g^{-1})^T$ where T is transpose. This is not a polynomial representation.
- Take $V = \mathbf{C}^2$ and $W = \mathbf{C}^3$ and ρ given by

$$\mathbf{GL}_2(\mathbf{C}) \to \mathbf{GL}_3(\mathbf{C})$$
$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} g_{1,1}^2 & g_{1,1}g_{1,2} & g_{1,2}^2 \\ 2g_{1,1}g_{2,1} & g_{1,1}g_{2,2} + g_{1,2}g_{2,1} & 2g_{1,2}g_{2,2} \\ g_{2,1}^2 & g_{2,1}g_{2,2} & g_{2,2}^2 \end{pmatrix}$$

- Take $V = \mathbf{C}^n$ and $W = \mathbf{C}$ and $\rho(g) = \det(g)$. More generally, for any integer d, we can take $\rho(g) = \det(g)^d$. This is a polynomial representation if and only if $d \geq 0$.

- For a non-example, take $V = W = \mathbf{C}^n$. Take $\rho(g) = \overline{g}$ where $\overline{g}$ means take the complex conjugate of each entry. Then $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}_n(\mathbf{C})$ is a homomorphism, but it is not algebraic. $\square$

We will think of ρ as giving us an action of $\mathbf{GL}(V)$ on W, i.e., if $g \in \mathbf{GL}(V)$ and $w \in W$, we define $g \cdot w = \rho(g)(w)$. Then this is a group action in the usual sense. Sometimes we will call W the representation.

Here are some general facts. First, pick a maximal torus $T \subset \mathbf{GL}(V)$. If $\rho\colon \mathbf{GL}(V) \to \mathbf{GL}(W)$ is a representation, then there is a basis $w_1, \dots, w_r$ for W such that each w_i is an eigenvector of $\rho(t)$ for all $t \in T$. Pick an ordered basis for V so that T is the set of diagonal matrices. If the entries of t are $x_1, \dots, x_n$, then with respect to the basis $w_1, \dots, w_r$, $\rho(t)$ is diagonal and the entries are of the form $x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$ for some $\mu_{i,1}, \dots, \mu_{i,n} \in \mathbf{Z}$ (why?). Any vector w which is an eigenvector for all $\rho(t)$ is called a **weight vector** and $(\mu_1, \dots, \mu_n)$ is its **weight** if $\rho(t)(w) = x_1^{\mu_1} \cdots x_n^{\mu_n} w$ for all $x_1, \dots, x_n$. We also write $\mu(t)$ for $x_1^{\mu_1} \cdots x_n^{\mu_n}$.

The **character** of ρ is defined to be the function $\mathrm{char}(\rho)(x_1, \dots, x_n) = \mathrm{Tr}\, \rho(t)$ where t is the diagonal matrix with entries $x_1, \dots, x_n$ and Tr denotes trace. Alternatively, $\mathrm{char}(\rho)(x_1, \dots, x_n) = \sum_{i=1}^r x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$.

Example 1.1.2. We compute the characters from Example 1.1.1.

- The basis $w_1, \dots, w_n$ is already an eigenbasis, so the character is $x_1 + \cdots + x_n$.
- Again, the standard basis is an eigenbasis, so the character is $x_1^{-1} + \cdots + x_n^{-1}$.
- Taking g to be diagonal with entries x_1, x_2, we see that $\rho(g)$ is also diagonal and its trace is $x_1^2 + x_1x_2 + x_2^2$.
- The character is $(x_1 \cdots x_n)^d$. $\square$

Basic operations transform easily on the level of characters:

- If $\rho_i\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_i)$ are representations for $i = 1, 2$, we can form the direct sum representation $\rho_1 \oplus \rho_2\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_1 \oplus V_2)$ via
$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$
and
$$\mathrm{char}(\rho_1 \oplus \rho_2)(x_1, \dots, x_n) = \mathrm{char}(\rho_1)(x_1, \dots, x_n) + \mathrm{char}(\rho_2)(x_1, \dots, x_n).$$
In terms of group action, this is given by $g \cdot (v_1, v_2) = (g \cdot v_1, g \cdot v_2)$.
- If $\rho_i\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_i)$ are representations for $i = 1, 2$, we can form the tensor product representation $\rho_1 \otimes \rho_2\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V_1 \otimes V_2)$ via (assuming $\rho_1(g)$ is $N \times N$):
$$(\rho_1 \otimes \rho_2)(g) = \begin{pmatrix} \rho_1(g)_{1,1}\rho_2(g) & \rho_1(g)_{1,2}\rho_2(g) & \cdots & \rho_1(g)_{1,N}\rho_2(g) \\ \rho_1(g)_{2,1}\rho_2(g) & \rho_1(g)_{2,2}\rho_2(g) & \cdots & \rho_1(g)_{2,N}\rho_2(g) \\ \vdots & & & \\ \rho_1(g)_{N,1}\rho_2(g) & \rho_1(g)_{N,2}\rho_2(g) & \cdots & \rho_1(g)_{N,N}\rho_2(g) \end{pmatrix}$$
(here we are multiplying $\rho_2(g)$ by each entry of $\rho_1(g)$ and creating a giant block matrix) and
$$\mathrm{char}(\rho_1 \otimes \rho_2)(x_1, \dots, x_n) = \mathrm{char}(\rho_1)(x_1, \dots, x_n) \cdot \mathrm{char}(\rho_2)(x_1, \dots, x_n).$$
In terms of group action, this is given by $g \cdot \sum_i v_i \otimes w_i = \sum_i g \cdot v_i \otimes g \cdot w_i$.

- If $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V)$ is a representation, then we have an action on the dual space V^* as follows. Given a linear functional $f\colon V \to \mathbf{C}$, we define $g \cdot f$ to be the linear functional given by $(g \cdot f)(v) = f(g^{-1} \cdot v)$. This gives a representation $\rho^\vee\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(V^*)$. In terms of matrices, if we pick an ordered basis for V and use the dual basis for V^*, we have $\rho^\vee(g) = (\rho(g)^{-1})^T$ where T denotes transpose.

  The character is given by $\mathrm{char}(\rho^\vee)(x_1, \dots, x_n) = \mathrm{char}(\rho)(x_1^{-1}, \dots, x_n^{-1})$.

Given two representations W and W' of $\mathbf{GL}(V)$, a **homomorphism** between them is a linear map $f\colon W \to W'$ such that $f(g \cdot w) = g \cdot f(w)$ for all $g \in \mathbf{GL}(V)$ and $w \in W$. We say that W and W' are **isomorphic** if there is a homomorphism between them which is an invertible linear map.

A **subrepresentation** of W is a subspace $U \subset W$ such that $g \cdot u \in U$ for all $u \in U$ and $g \in \mathbf{GL}(V)$. A nonzero representation W is **irreducible** (or **simple**) if its only subrepresentations are either 0 or W. A representation is **semisimple** if it is isomorphic to a direct sum of simple representations.

Theorem 1.1.3. *(1) Every finite-dimensional representation of* $\mathbf{GL}(V)$ *is semisimple.*

(2) Two representations of $\mathbf{GL}(V)$ *are isomorphic if and only if they have the same character.*

(3) (Schur's lemma) There are no nonzero homomorphisms between non-isomorphic simple representations. Any homomorphism from a simple representation to itself must be a scalar multiple of the identity.

Let $B \subset \mathbf{GL}(V)$ be a Borel subgroup which contains our maximal torus T. A nonzero vector $w \in W$ is a **highest weight vector** if $b \cdot w$ is a scalar multiple of w for all $b \in B$. In particular, it is an eigenvector for all $\rho(t)$ for $t \in T$, so it is also a weight vector. If μ is the weight of this vector, we write $\mu(b)$ for the scalar multiple i.e., $b \cdot w = \mu(b)w$.

Example 1.1.4. We discuss the examples from Example 1.1.1.

- The standard basis vectors $w_1, \dots, w_n$ are the weight vectors. The weight of w_i is $(0, \dots, 1, \dots, 0)$ with a 1 in position i. The only highest weight vector is w_1 with weight $(1, 0, \dots, 0)$.
- The standard basis vectors $w_1, \dots, w_n$ are the weight vectors. The weight of w_i is $(0, \dots, -1, \dots, 0)$ with a -1 in position i. The only highest weight vector is w_n with weight $(0, \dots, 0, -1)$.
- The standard basis vectors e_1, e_2, e_3 of $\mathbf{C}^3$ are weight vectors with weights $(2, 0)$, $(1, 1)$, and $(0, 2)$. The only highest weight vector is e_1 with weight $(2, 0)$.
- $1 \in \mathbf{C}$ is a weight vector with weight $(d, \dots, d)$ and is also a highest weight vector. $\square$

Theorem 1.1.5.
- *Every finite-dimensional representation contains a highest weight vector.*
- *Any two highest weight vectors (for a particular choice of Borel subgroup) in an irreducible representation are scalar multiples of each other.*
- *The weight* $(\mu_1, \dots, \mu_n)$ *of a highest weight vector satisfies* $\mu_1 \geq \cdots \geq \mu_n$.
- *For every* $(\mu_1, \dots, \mu_n) \in \mathbf{Z}^n$ *satisfying* $\mu_1 \geq \cdots \geq \mu_n$*, there is an irreducible representation whose highest weight vector has that weight. Furthermore, this representation is unique up to isomorphism and the representation is polynomial if and only if* $\mu_n \geq 0$.

In particular, we conclude that the irreducible representations are determined by their highest weight vectors, and if we want to express an arbitrary representation as a direct sum of irreducible representations, it suffices to find all highest weight vectors (why?). We use $\mathbf{S}_\lambda(V)$ to denote an irreducible representation whose highest weight is λ. We will see different ways to construct this representation.

Note that tensoring a representation by the dth power of the determinant representation adds d to all components of each weight, and in particular for highest weights. So the result above implies the following, for which we will give an independent proof:

Proposition 1.1.6. *If W is a rational representation, then there exists d such that $W \otimes \det^d$ is a polynomial representation.*

Proof. With respect to some choice of basis, the entries of $\rho\colon \mathbf{GL}_n(\mathbf{C}) \to \mathbf{GL}(W)$ are rational functions $a(x)/b(x)$ of the matrix entries $x_{i,j}$, such that $b(g) \neq 0$ whenever g is invertible, so it suffices to prove that for every such rational function, b is a power of the determinant function (up to a scalar). We prove this by induction on $\deg b$. If $\deg b = 0$, then it is a constant and there is nothing to say.

Otherwise, since $\mathbf{C}$ is algebraically closed, the polynomial b has a zero somewhere on the space of all matrices. By assumption, all of its zeroes are singular matrices, so that $b(x) = 0$ implies that $\det(x) = 0$. The next result shows that det is an irreducible polynomial, so that det divides b. But then $b' = b/\det$ is another polynomial of lower degree such that $b'(g) \neq 0$ whenever g is invertible, so by induction, b' is a scalar times a power of the determinant. □

Lemma 1.1.7. *As a polynomial in the n variables $x_{i,j}$,* det *is irreducible.*

Proof. Suppose we have a factorization $\det = \alpha\beta$. Note that det has degree 1 in each variable $x_{i,j}$ separately, so for each $x_{i,j}$, it must be that either α has degree 1 and β has degree 0 with respect to $x_{i,j}$, or the other way around. Now consider $x_{i,j}$ and $x_{i',j}$ together. Since no term of det involves both at the same time, if α has degree 1 in $x_{i,j}$, then it must also have degree 1 in $x_{i',j}$ (if not, then we can write $\alpha = \alpha_0 x_{i,j} + \alpha_1$ and $\beta = \beta_0 x_{i',j} + \beta_1$ where none of $\alpha_0, \alpha_1, \beta_0, \beta_1$ involve $x_{i,j}$ or $x_{i',j}$ and multiplying it out gives a contradiction). The same is true if we consider two variables $x_{i,j}$ and $x_{i,j'}$ together. This implies that if α has degree 1 in one of the variables, then it has degree 1 in all of the variables, i.e., β is a constant. Otherwise, the same reasoning implies that β has degree 1 in all of the variables and α is a constant. □

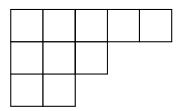
1.1.1. *Partitions.* A **partition** of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ and $\lambda_1 + \cdots + \lambda_k = n$. We will consider two partitions the same if their nonzero entries are the same. It will also be convenient to make the convention that $\lambda_i = 0$ whenever $i > \ell(\lambda)$. And for shorthand, we may omit the commas, so the partition $(1, 1, 1, 1)$ of 4 can be written as 1111. As a further shorthand, the exponential notation is used for repetition, so for example, 1^4 is the partition $(1, 1, 1, 1)$. We let $\mathrm{Par}(n)$ be the set of partitions of n, and denote the size by $p(n) = |\mathrm{Par}(n)|$. By convention, $\mathrm{Par}(0)$ consists of exactly one partition, the empty one.

Example 1.1.8.

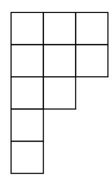
$$\begin{aligned}
\mathrm{Par}(1) &= \{1\},\\
\mathrm{Par}(2) &= \{2, 1^2\},\\
\mathrm{Par}(3) &= \{3, 21, 1^3\},\\
\mathrm{Par}(4) &= \{4, 31, 22, 21^2, 1^4\},\\
\mathrm{Par}(5) &= \{5, 41, 32, 31^2, 2^2 1, 21^3, 1^5\}.
\end{aligned}$$ $\square$

If λ is a partition of n, we write $|\lambda| = n$ (**size**). Also, $\ell(\lambda)$ is the number of nonzero entries of λ (**length**). For each i, $m_i(\lambda)$ is the number of entries of λ that are equal to i.

It will often be convenient to represent partitions graphically. This is done via **Young diagrams** $Y(\lambda)$, which is a collection of left-justified boxes with λ_i boxes in row i. For example, the Young diagram



corresponds to the partition $(5, 3, 2)$. Flipping across the main diagonal gives another partition $\lambda^\dagger$, called the **transpose**. In our example, flipping gives



So $(5, 3, 2)^\dagger = (3, 3, 2, 1, 1)$. In other words, the role of columns and rows has been interchanged. This is an important involution of $\mathrm{Par}(n)$ which we will use later.

We use $\nu \subseteq \lambda$ to mean that $\nu_i \le \lambda_i$ for all i. In that case, $Y(\nu)$ is a subset of $Y(\lambda)$ and we use λ/ν to denote the shape $Y(\lambda) \setminus Y(\nu)$.

1.1.2. $\mathbf{GL}_n \times \mathbf{GL}_m$. We'd also like to consider representations of a product of general linear groups $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$. Much of the previous discussion applies. The Borel subgroup gets replaced with the product $B \times B'$ where $B \subset \mathbf{GL}_n(\mathbf{C})$ and $B' \subset \mathbf{GL}_m(\mathbf{C})$ are the upper-triangular matrices in each group. Similarly, the maximal torus is replaced by the product $T \times T'$. Weights are now pairs of vectors $(\mu_1, \dots, \mu_n), (\mu'_1, \dots, \mu'_m)$ and irreducible representations now correspond to highest weights which satisfy $\mu_1 \ge \cdots \ge \mu_n$ and $\mu'_1 \ge \cdots \ge \mu'_m$. The corresponding irreducible is denoted $\mathbf{S}_\mu(\mathbf{C}^n) \otimes \mathbf{S}_{\mu'}(\mathbf{C}^m)$.

1.2. Multiplicity-free actions.

This section is based on [Ho].

From the last part, we see that any representation W of $\mathbf{GL}_n(\mathbf{C})$ is isomorphic to $\bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n)^{\oplus m_\lambda}$ for some $m_\lambda \ge 0$. The integer m_λ is the **multiplicity** of $\mathbf{S}_\lambda(\mathbf{C}^n)$ in W. We say that W is **multiplicity-free** if $m_\lambda \le 1$ for all λ. Similarly, any representation W of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ is isomorphic to $\bigoplus_{\lambda,\lambda'} (\mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{\lambda'}(\mathbf{C}^m))^{\oplus m_{\lambda,\lambda'}}$ and it is multiplicity-free if $m_{\lambda,\lambda'} \le 1$ for all pairs λ, λ'.

Given a vector space U, we let $\mathrm{Sym}^d U^*$ denote the symmetric power of the dual space U^*. This is the quotient of $(U^*)^{\otimes d}$ by the subspace spanned by expressions $u_1 \otimes \cdots \otimes u_d - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)}$ over all choices of permutations σ and $u_1, \dots, u_d \in U^*$. If we pick a basis $v_1, \dots, v_n$ for U, then we get a dual basis $x_1, \dots, x_n$ for U^* and then $\mathrm{Sym}^d U^*$ is the space of degree d homogeneous polynomials in $x_1, \dots, x_n$ and has a basis given by all degree d

monomials in the x_i. If U is a representation of $\mathbf{GL}(V)$, then so is $\mathrm{Sym}^d U^*$. We define $\mathrm{Sym}\, U^* = \bigoplus_{d \ge 0} \mathrm{Sym}^d U^*$, which is the space of all polynomials in $x_1, \dots, x_n$. These are functions on U, so that given $f \in \mathrm{Sym}\, U^*$ and $u \in U$, the evaluation $f(u)$ makes sense. This is an infinite-dimensional representation of $\mathbf{GL}(V)$, but it is a direct sum of finite-dimensional ones, so everything we have said so far still applies.

Example 1.2.1. Consider the case $U = V^*$ so that $U^* = V$. I claim that $\mathrm{Sym}^d V$ is irreducible for any $d \ge 0$. Pick a basis $x_1, \dots, x_n$ for V and let T, B be the subgroups of diagonal and upper-triangular matrices with respect to this basis. Then the weight vectors are the monomials in $x_1, \dots, x_n$ and x_1^d is the only one which is a highest weight vector. So $\mathrm{Sym}^d V$ is irreducible with highest weight $(d, 0, \dots, 0)$. We see that $\mathrm{Sym}\, V$ is multiplicity-free. □

We'd like to give a general criteria for $\mathrm{Sym}\, U^*$ to be multiplicity-free for a finite-dimensional representation U. This requires some discussion of the Zariski topology.

1.2.1. *Zariski topology.* Let U be a vector space. A function $f\colon U \to \mathbf{C}$ is polynomial if it can be written as a polynomial with respect to some (and hence, any) basis for U. Given a (possibly infinite) set of polynomials I, we define $Z(I) \subset U$ to be the common solution set, i.e., $Z(I) = \{u \in U \mid f(u) = 0 \text{ for all } f \in I\}$. Note that if $f, g \in I$ and h is any polynomial, then $Z(I) = Z(I \cup \{f+g\}) = Z(I \cup \{fh\})$. For that reason, we will always assume that I is closed under addition and multiplication by arbitrary polynomials, i.e., that I is an **ideal**. Given a set of polynomials $\{f_i\}$, we let $\langle f_i \rangle$ denote the smallest ideal containing the f_i, i.e., $\langle f_i \rangle = \{\sum_j h_j f_j\}$ for all finite sums. We say that I is **generated** by $\{f_i\}$ if $I = \langle f_i \rangle$.

An important fundamental fact:

Theorem 1.2.2 (Hilbert basis theorem)**.** *Every ideal can be generated by a finite set of polynomials.*

The sum of two ideals is $I_1 + I_2 = \{f + g \mid f \in I_1,\ g \in I_2\}$ and the product $I_1 I_2$ is the ideal generated by $\{fg \mid f \in I_1,\ g \in I_2\}$. Infinite sums of ideals make sense: it is the ideal generated by all finite sums of elements in the ideals, but infinite products generally do not make sense.

The **Zariski topology** on U is the topology whose closed sets are the subsets of the form $Z(I)$ for an ideal I. It's easy to check that this is a topology:

- Empty set is closed: $\varnothing = Z(\langle 1 \rangle)$
- U is closed: $U = Z(0)$
- The intersection of closed sets is closed: $\bigcap_j Z(I_j) = Z(\sum_j I_j)$
- A finite union of closed sets is closed: $Z(I_1) \cup \cdots \cup Z(I_r) = Z(I_1 \cdots I_r)$

Closed subsets of U (with the subspace topology) are called **affine varieties**. The coordinate ring of an affine variety $X \subset U$ is denoted $\mathbf{C}[X]$ and is the quotient of $\mathrm{Sym}\, U$ by the ideal of all polynomials which are identically 0 on X. Note that $\mathbf{C}[U] = \mathrm{Sym}\, U^*$.

Open subsets of affine varieties (with the subspace topology) are called **quasi-affine varieties**. Every affine variety is quasi-affine but the converse does not generally hold.

Example 1.2.3. Note that $\mathbf{GL}_n(\mathbf{C})$ can be interpreted as a quasi-affine variety: it is a subset of $\mathbf{C}^{n^2}$ in the obvious way and its complement is the set of points whose determinant is 0. Since the determinant is a polynomial function in n^2 variables, we see that $\mathbf{GL}_n(\mathbf{C})$ is an open subset.

Actually, we can realize it as an affine variety if we add 1 more variable. Consider $\mathbf{C}^{n^2+1}$ and let t be the extra coordinate. Then

$$Z(\langle t\det -1\rangle) = \{(g,\lambda) \mid g \in \mathbf{GL}_n(\mathbf{C}),\ \lambda = 1/\det(g)\}.$$

The extra information λ is not really extra, so the projection to the first n^2 coordinates identifies $Z(\langle t\det -1\rangle)$ with $\mathbf{GL}_n(\mathbf{C})$. This is really an isomorphism of varieties, but we will not go into the details of what that means.

Just for reference, this means that $\mathbf{GL}_n(\mathbf{C})$ is an **algebraic group**: an affine variety which has a group structure (such that the group product and inverse can be expressed by rational functions). $\square$

A topological space X is **irreducible** if, whenever $X = X_1 \cup X_2$ with both X_1, X_2 closed subsets, we must have $X_1 = X$ or $X_2 = X$.

Proposition 1.2.4. *A vector space U with the Zariski topology is an irreducible topological space.*

Proof. Suppose that $U = Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$ for ideals I_1, I_2. Then $I_1 I_2$ must be the 0 ideal: if there is a nonzero polynomial f in the product, then there is some point $u \in U$ such that $f(u) \neq 0$. But this means that all pairwise products fg are 0 where $f \in I_1$ and $g \in I_2$. Since the product of nonzero polynomials is nonzero, this means that either $I_1 = 0$ or $I_2 = 0$, i.e., that $Z(I_1) = U$ or $Z(I_2) = U$. $\square$

Proposition 1.2.5. *Every non-empty open subset Y of an irreducible space X is dense, i.e., if $X' \supset Y$ is a closed subset of X, then $X = X'$.*

Proof. With notation as in the statement, we have $X = X' \cup (X \setminus Y)$. Since X is irreducible, either $X' = X$ or $X \setminus Y = X$. The latter means that Y is empty, so it must be that $X' = X$. $\square$

Now we come back to multiplicity-free spaces. Given an affine variety X in a representation U of $\mathbf{GL}(V)$ and a subgroup $G \subset \mathbf{GL}(V)$, we define the G-orbits of X to be the equivalence classes of the equivalence relation on X given by $u \sim u'$ if $u' = g \cdot u$ for some $g \in G$.

Theorem 1.2.6. *Let X be an affine variety in some representation of $\mathbf{GL}(V)$ and let B be a Borel subgroup. Suppose that there is a B-orbit Y on X which is dense. Then*

(a) *$\mathbf{C}[X]$ is a multiplicity-free representation.*

(b) *Let λ be the weight of a highest weight vector of $\mathbf{C}[X]$. Pick $u \in Y$ and let H be the stabilizer of u, i.e., $H = \{h \in \mathbf{GL}(V) \mid h \cdot u = u\}$. Then $\lambda(h) = 1$ for all $h \in H \cap B$.*

Proof. (a) Pick two highest weight vectors $f, g \in \mathbf{C}[X]$ with the same weight λ. Pick a point $u \in Y$. Note that $f(b \cdot u) = (b^{-1} \cdot f)(u) = \lambda(b^{-1}) f(u)$. If $f(u) = 0$, then f is 0 on all of Y, which means it is the zero polynomial since $Z(f) \supset Y$. Similarly, $g(u) \neq 0$, so there is a nonzero scalar α so that $g(u) = \alpha f(u)$. But then $g - \alpha f$ is a highest weight vector of weight λ, but $(g - \alpha f)(u) = 0$, so the previous reasoning shows that $g - \alpha f$ is the zero polynomial, i.e., that f, g must be scalar multiples of each other.

(b) Continuing the same notation, if $h \in H \cap B$, then $f(u) = f(h \cdot u) = \lambda(h^{-1}) f(u)$. Since $f(u) \neq 0$, this means $\lambda(h^{-1}) = 1$. Since $H \cap B$ is a group and h was arbitrary, this implies $\lambda(h) = 1$ for all $h \in H \cap B$. $\square$

Finally, here's a useful observation.

Proposition 1.2.7. *Let X be an irreducible affine variety in some representation of $\mathbf{GL}(V)$. If λ and μ are highest weights for irreducible representations appearing in $\mathbf{C}[X]$, then so is $\lambda+\mu$.*

Proof. Let f and g be highest weight vectors of weights λ and μ in $\mathbf{C}[X]$. We claim that $fg \neq 0$. If so, then $Z(f) \cup Z(g) = X$ which means that $X = Z(f)$ or $X = Z(g)$ since X is irreducible. But both f and g are nonzero so this is impossible. Next, for any $b \in B$, we have $b \cdot (fg) = (b \cdot f)(b \cdot g) = \lambda(b)\mu(b)fg = (\lambda+\mu)(b)fg$. □

These results apply just as well if we have a product of general linear groups with the product of Borel subgroups replacing B.

1.2.2. *Example: generic matrices.* Pick integers $m \geq n$. We consider the product $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ with the representation $U = (\mathbf{C}^n \otimes \mathbf{C}^m)^*$ being the space of $n \times m$ matrices. The action is given by

$$(g,h) \cdot u = (g^{-1})^T u h^{-1}.$$

Let J be the $n \times m$ matrix with $J_{i,i} = 1$ for $i = 1, \dots, n$ and 0's elsewhere. Let $B \subset \mathbf{GL}_n(\mathbf{C})$ be the subgroup of upper-triangular matrices, and similarly, let $B' \subset \mathbf{GL}_m(\mathbf{C})$ be the subgroup of upper-triangular matrices.

Let A_i be the upper-left $i \times i$ submatrix of the generic matrix $\varphi = (\varphi_{ij})$ and let $f_i = \det A_i$.

Proposition 1.2.8. *f_i is a highest weight vector with weight $(1,\dots,1,0,\dots,0), (1,\dots,1,0,\dots,0)$ (the number of 1's in each vector is i).*

Proof. Pick upper-triangular matrices $g \in \mathbf{GL}_n(\mathbf{C})$ and $h \in \mathbf{GL}_m(\mathbf{C})$ and write $g = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$ and $h = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix}$ where x_1, x_2 are $i \times i$. Then $(g,h) \cdot f_i$ is the determinant of the upper-left $i \times i$ submatrix of $g^T \varphi h$, which is $\det(x_1^T A_i x_2) = \det(x_1)\det(x_2) f_i$. In particular, it is a highest weight vector. If g, h are diagonal, then $\det(x_1)\det(x_2)$ is just the product of the first i entries of each of g and h, so we get the weight also. □

Lemma 1.2.9. *The $B \times B'$ orbit containing J is open and dense.*

Proof. We claim that the orbit is precisely the set of matrices A such that $f_i(A) \neq 0$ for $i = 1, \dots, n$. It is easy to see that being in the orbit implies the condition on submatrices, so we just prove the reverse direction. Before proving the claim, note that this shows that the orbit is $\bigcap_{i=1}^n (U \setminus Z(f_i))$, which is open (denseness follows from Propositions 1.2.4 and 1.2.5).

We first handle the case $n = m$. Then J is just the identity matrix, so the $B \times B'$ orbit of J is the set of matrices A with an LU factorization, i.e., $A = LU$ where L is lower-triangular and invertible with 1's on the diagonal, and U is upper-triangular and invertible.

We proceed by induction on n. The case $n = 1$ is immediate, so for the general case, write $A = \begin{bmatrix} A' & b \\ c & d \end{bmatrix}$ where A' is $(n-1) \times (n-1)$. Then all of the upper left submatrices of A' are invertible and so we have a factorization $A' = L'U'$. Then we have

$$\begin{bmatrix} A' & b \\ c & d \end{bmatrix} = \begin{bmatrix} L' & 0 \\ x & 1 \end{bmatrix} \begin{bmatrix} U' & y \\ 0 & z \end{bmatrix}$$

where $y = (L')^{-1}b$, $x = c(U')^{-1}$, and $z = d - xy$. We let L and U be these last two matrices. Then L is lower-triangular and invertible with 1's on the diagonal and U is upper-triangular and invertible (since $U = L^{-1}A$).

For the general case $m \ge n$, write $A = \begin{bmatrix} A' & A'' \end{bmatrix}$ where A' is $n \times n$ and its upper left submatrices are invertible. Then we can factor it as $A' = LU$ as above. Then we have

$$A = LJ \begin{bmatrix} U & L^{-1}A'' \\ 0 & \mathrm{id}_{m-n} \end{bmatrix}. \qquad \square$$

Lemma 1.2.10. *If (λ, λ') is the weight of a highest weight vector in $\operatorname{Sym} U^*$, then $\lambda_i = \lambda'_i$ for $1 \le i \le n$ and $\lambda'_j = 0$ for $j > n$. Also, $\lambda_n \ge 0$.*

Proof. The stabilizer of J contains pairs of diagonal matrices (g, h) where the entries of h are $x_1, \dots, x_m$ and the entries of g are $x_1^{-1}, \dots, x_n^{-1}$. By Theorem 1.2.6, we see that $x_1^{\lambda'_1 - \lambda_1} \cdots x_n^{\lambda'_n - \lambda_n} x_{n+1}^{\lambda'_{n+1}} \cdots x_m^{\lambda'_m} = 1$ for all $x_1, \dots, x_m$. This forces all of the exponents to be 0.

Finally, all weights attached to weight vectors in $\operatorname{Sym} U^*$ are non-negative. This follows from the formula for the action on U. $\square$

Finally, each pair (λ, λ') as above is the highest weight for some irreducible representation in $\operatorname{Sym} U^*$ by Proposition 1.2.7 since every partition λ is a sum of vectors of the form $(1, 1, \dots, 1, 0, \dots, 0)$.

Corollary 1.2.11 (Cauchy identity)**.** *We have an isomorphism of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ representations*

$$\operatorname{Sym}(\mathbf{C}^n \otimes \mathbf{C}^m) \cong \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_\lambda(\mathbf{C}^m)$$

where the sum is over all integer partitions $\lambda_1 \ge \cdots \lambda_n \ge 0$ and the second λ is understood to have $m - n$ 0's added at the end.

1.2.3. *Example: symmetric matrices.* The space U of $n \times n$ symmetric matrices has an action of $\mathbf{GL}_n(\mathbf{C})$ via

$$g \cdot X = (g^{-1})^T X g^{-1}.$$

Let f_1 be the function which takes the $(1,1)$-entry of a symmetric matrix. If g is upper-triangular, then $(g \cdot f_1)(X) = f_1(g^{-1} \cdot X) = g_{1,1}^2 f_1(X)$, so f_1 is a highest weight vector of U^* with highest weight $(2, 0, \dots, 0)$. So U^* contains a copy of $\operatorname{Sym}^2 \mathbf{C}^n$ as a $\mathbf{GL}_n(\mathbf{C})$-representation. Since they have the same dimension, $U^* \cong \operatorname{Sym}^2 \mathbf{C}^n$.

We want to repeat our analysis from the previous section to this new example.

Let f_i be the determinant of the upper-left $i \times i$ submatrix of a symmetric matrix.

Proposition 1.2.12. *f_i is a highest weight vector with weight $(2, \dots, 2, 0, \dots, 0)$ (the number of 2's is i).*

Proof. Let $g \in \mathbf{GL}_n(\mathbf{C})$ be an upper triangular matrix and write it as $g = \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}$ where g_1 is $i \times i$. Let X be a symmetric matrix and write it as $X = \begin{bmatrix} A & B \\ B^T & C \end{bmatrix}$ where A is $i \times i$. Then $(g \cdot f_i)(X) = f_i(g^{-1} \cdot X) = f_i(g^T X g)$. Then the upper left $i \times i$ submatrix of $g^T X g$ is $g_1^T A g_1$, and its determinant is $\det(g_1)^2 f_i(X)$, which shows that f_i is a highest weight vector of weight $(2, \dots, 2, 0, \dots, 0)$. $\square$

Lemma 1.2.13. *The B-orbit of the identity matrix I is open and dense.*

Proof. The B-orbit of I is the set of symmetric matrices which can be factored as $g^T g$ for some invertible upper triangular matrix g. We claim that this is the set of symmetric matrices X such that $f_i(X) \neq 0$ for $i = 1, \dots, n$, and prove it by induction on n.

If $n = 1$, this is clear. Otherwise, write $X = \begin{bmatrix} X' & y \\ y^T & z \end{bmatrix}$ where X' has size $(n-1) \times (n-1)$. By induction, $f_i(X') \neq 0$ for $i = 1, \dots, n-1$ and so we have $X' = h^T h$ for an invertible upper triangular $(n-1) \times (n-1)$ matrix h. Then we have

$$\begin{bmatrix} X' & y \\ y^T & z \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ y^T h^{-1} & \alpha \end{bmatrix} \begin{bmatrix} h & (h^T)^{-1} y \\ 0 & \alpha \end{bmatrix}$$

where $\alpha^2 = z - y^T h^{-1} (h^T)^{-1} y$. The new matrices we produced are automatically invertible since their product is invertible (by the assumption $f_n(X) \neq 0$).

This implies that the orbit is Zariski open (and hence dense). □

Lemma 1.2.14. *If λ is a weight of a highest weight vector in $\operatorname{Sym} U^*$, then λ_i is even for all i and $\lambda_n \geq 0$.*

Proof. We use Theorem 1.2.6. Let h be the diagonal matrix with 1's on the diagonal except for a -1 in position i. Then $h^T h = I$, and $\lambda(h) = (-1)^{\lambda_i}$, so λ_i is even. Since all weights of $\operatorname{Sym}(U^*)$ are non-negative, we also get that $\lambda_n \geq 0$. □

Corollary 1.2.15. *We have an isomorphism of $\mathbf{GL}_n(\mathbf{C})$ representations*

$$\operatorname{Sym}(\operatorname{Sym}^2 \mathbf{C}^n) \cong \bigoplus_\lambda \mathbf{S}_{2\lambda}(\mathbf{C}^n)$$

where the sum is over all integer partitions $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

Proof. We have shown that $\operatorname{Sym}(\operatorname{Sym}^2 \mathbf{C}^n)$ is multiplicity-free and all highest weights are of the form 2λ for an integer partition λ. It remains to show they all appear in the decomposition, but we can get all of them by multiplying the functions $f_1, \dots, f_n$ with various multiplicities. □

1.2.4. *Example: skew-symmetric matrices.* Given a vector space V and a positive integer d, the **dth exterior power** $\bigwedge^d V$ is the quotient of $V^{\otimes d}$ by the subspace spanned by elements of the form $v_1 \otimes \cdots \otimes v_d$ where $v_i = v_j$ for some $i \neq j$. Note that the implies that swapping two elements introduces a sign in $\bigwedge^d V$, for example when $d = 2$ we have:

$$0 = (v_1 + v_2) \otimes (v_1 + v_2) = v_1 \otimes v_1 + v_1 \otimes v_2 + v_2 \otimes v_1 + v_2 \otimes v_2 = v_1 \otimes v_2 + v_2 \otimes v_1$$

and the general case is similar (but with more cumbersome notation). The coset of $v_1 \otimes \cdots \otimes v_d$ is denoted $v_1 \wedge \cdots \wedge v_d$; from what we said it satisfies $v_1 \wedge \cdots \wedge v_d = (\operatorname{sgn} \sigma) v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(d)}$ for any permutation σ. If $e_1, \dots, e_r$ is a basis for V, then a basis for $\bigwedge^d V$ is given by $\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < \cdots < i_d \leq r\}$ so that it has dimension $\binom{r}{d}$.

The space U of $n \times n$ skew-symmetric matrices has an action of $\mathbf{GL}_n(\mathbf{C})$ via

$$g \cdot X = (g^{-1})^T X g^{-1}.$$

Let f_1 be the function which takes the $(1,2)$-entry of a skew-symmetric matrix. If g is upper-triangular, then $(g \cdot f_1)(X) = f_1(g^{-1} \cdot X) = g_{1,1} g_{2,2} f_1(X)$, so f_1 is a highest weight vector of U^* with highest weight $(1, 1, 0, \dots, 0)$. So U^* contains a copy of $\bigwedge^2 \mathbf{C}^n$ as a $\mathbf{GL}_n(\mathbf{C})$-representation. Since they have the same dimension, $U^* \cong \bigwedge^2 \mathbf{C}^n$.

We want to repeat our analysis from the previous section to this new example. Before doing this, we need a short digression on Pfaffians.

First, given a skew-symmetric $n \times n$ matrix X, we have $\det(X) = \det(X^T) = \det(-X) = (-1)^n \det(X)$, so $\det X = 0$ if n is odd. In particular, the rank of a skew-symmetric matrix is always even. Let X have size $2i \times 2i$. Let Π be the set of permutations σ of $2i$ that satisfy $\sigma(1) < \sigma(3) < \cdots < \sigma(2i-1)$ and $\sigma(1) < \sigma(2)$, $\sigma(3) < \sigma(4)$, ..., $\sigma(2i-1) < \sigma(2i)$. The **Pfaffian** of X is defined by

$$\mathrm{Pf}(X) = \sum_{\sigma \in \Pi} (\operatorname{sgn} \sigma) X_{\sigma(1),\sigma(2)} X_{\sigma(3),\sigma(4)} \cdots X_{\sigma(2i-1),\sigma(2i)}.$$

The formula is not so important, but here are two important properties (whose proofs we will omit):

- $(\mathrm{Pf}\, X)^2 = \det X$
- $\mathrm{Pf}(gXg^T) = (\det g)(\mathrm{Pf}\, X)$ for any $g \in \mathbf{GL}_{2i}(\mathbf{C})$.

Let f_i be the Pfaffian of the upper-left $2i \times 2i$ submatrix of a skew-symmetric matrix. It is a polynomial function of degree i.

Proposition 1.2.16. *f_i is a highest weight vector with weight $(1, \dots, 1, 0, \dots, 0)$ (the number of 1's is $2i$).*

Proof. Let $g \in \mathbf{GL}_n(\mathbf{C})$ be an upper triangular matrix and write it as $g = \begin{bmatrix} g_1 & g_2 \\ 0 & g_3 \end{bmatrix}$ where g_1 is $i \times i$. Let X be a skew-symmetric matrix and write it as $X = \begin{bmatrix} A & B \\ -B^T & C \end{bmatrix}$ where A is $2i \times 2i$. Then $(g \cdot f_i)(X) = f_i(g^{-1} \cdot X) = f_i(g^T X g)$. The upper left $2i \times 2i$ submatrix of $g^T X g$ is $g_1^T A g_1$, and its Pfaffian is $\det(g_1) f_i(X)$, which shows that f_i is a highest weight vector of weight $(1, \dots, 1, 0, \dots, 0)$. $\square$

Consider the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. If n is even, let J_n be the block diagonal matrix consisting of $n/2$ copies of this matrix. If n is odd, then let J_n be the block diagonal matrix consisting of $(n-1)/2$ copies of this matrix and one extra 0 at the end.

Lemma 1.2.17. *The B-orbit of the identity matrix J_n is open and dense.*

Proof. The B-orbit of J_n is the set of skew-symmetric matrices which can be factored as $g^T J_n g$ for some invertible upper triangular matrix g. We claim that this is the set of skew-symmetric matrices X such that $f_i(X) \neq 0$ for $i = 1, \dots, \lfloor n/2 \rfloor$, and prove it by induction on n.

We first handle the case when n is even. If $n = 2$, this is clear. Otherwise, write $X = \begin{bmatrix} X' & y \\ -y^T & z \end{bmatrix}$ where X' has size $(n-2) \times (n-2)$. By induction, $f_i(X') \neq 0$ for $i = 1, \dots, (n-2)/2$ and so we have $X' = h^T J_{n-2} h$ for an invertible upper triangular $(n-2) \times (n-2)$ matrix h. Then we have

$$\begin{bmatrix} X' & y \\ -y^T & z \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ \alpha^T & \beta^T \end{bmatrix} \begin{bmatrix} J_{n-2} & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} h & \alpha \\ 0 & \beta \end{bmatrix}$$

where $\alpha = (h^T J_{n-2})^{-1} y$ and $\beta = \begin{bmatrix} z_{1,2} & 0 \\ 0 & 1 \end{bmatrix}$. The new matrices we produced are automatically invertible since their product is invertible (by the assumption $f_{n/2}(X) \neq 0$).

Now we do the case when n is odd. Write $X = \begin{bmatrix} X' & y \\ -y^T & 0 \end{bmatrix}$ where X' has size $(n-1)\times(n-1)$. Since $n-1$ is even, we can factor $X' = h^T J_{n-1} h$ for an upper triangular invertible matrix h. Then we have

$$\begin{bmatrix} X' & y \\ -y^T & 0 \end{bmatrix} = \begin{bmatrix} h^T & 0 \\ \alpha^T & 1 \end{bmatrix} \begin{bmatrix} J_{n-1} & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} h & \alpha \\ 0 & 1 \end{bmatrix}$$

where $\alpha = (h^T J_{n-1})^{-1} y$ and clearly $\begin{bmatrix} h & \alpha \\ 0 & 1 \end{bmatrix}$ is invertible and upper-triangular.

This implies that the orbit is Zariski open (and hence dense). $\square$

Lemma 1.2.18. *If λ is a weight of a highest weight vector in $\operatorname{Sym} U^*$, then $\lambda_i^\dagger$ is even for all i and $\lambda_n \geq 0$.*

Proof. We use Theorem 1.2.6. If $n = 2m$ is even, let h be the diagonal matrix with entries $x_1, x_1^{-1}, \dots, x_m, x_m^{-1}$. Then $h^T J_n h = J_n$ and $\lambda(h) = x_1^{\lambda_1 - \lambda_2} x_2^{\lambda_3 - \lambda_4} \cdots x_m^{\lambda_{n-1} - \lambda_n}$. If this is 1 for all choices of $x_1, \dots, x_m$, then we must have $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, etc. This is the same as saying that $\lambda_i^\dagger$ is even for all i.

If $n = 2m+1$ is odd, let h be the diagonal matrix with entries $x_1, x_1^{-1}, \dots, x_m, x_m^{-1}, x_{m+1}$. Then $h^T J_n h = J_n$ and $\lambda(h) = x_1^{\lambda_1 - \lambda_2} x_2^{\lambda_3 - \lambda_4} \cdots x_m^{\lambda_{n-2} - \lambda_{n-1}} x_{m+1}^{\lambda_n}$. If this is 1 for all choices of $x_1, \dots, x_{m+1}$, then we must have $\lambda_1 = \lambda_2$, $\lambda_3 = \lambda_4$, etc. and $\lambda_n = 0$. This is the same as saying that $\lambda_i^\dagger$ is even for all i.

In either case, all weights of $\operatorname{Sym}(U^*)$ are non-negative, so we also get that $\lambda_n \geq 0$. $\square$

Corollary 1.2.19. *We have an isomorphism of $\mathbf{GL}_n(\mathbf{C})$ representations*

$$\operatorname{Sym}(\bigwedge^2 \mathbf{C}^n) \cong \bigoplus_\lambda \mathbf{S}_{(2\lambda)^\dagger}(\mathbf{C}^n)$$

where the sum is over all integer partitions such that $2\lambda_1 \leq n$.

Proof. We have shown that $\operatorname{Sym}(\bigwedge^2 \mathbf{C}^n)$ is multiplicity-free and all highest weights are of the form $(2\lambda)^\dagger$ for an integer partition λ. It remains to show they all appear in the decomposition, but we can get all of them by multiplying the functions $f_1, \dots, f_{\lfloor n/2 \rfloor}$ with various multiplicities. $\square$

1.3. **Schur functors.** This section will mostly give facts without proofs.

Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of d and set $\mu = \lambda^\dagger = (\mu_1, \dots, \mu_s)$. The **Schur functor** $\mathbf{S}_\lambda V$ is defined to be the image of the following composition:

$$\begin{aligned} \bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V &\to V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s} \\ &\to V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r} \\ &\to \operatorname{Sym}^{\lambda_1} V \otimes \cdots \otimes \operatorname{Sym}^{\lambda_r} V, \end{aligned}$$

where the first map is given by comultiplication:

$$\begin{aligned} \bigwedge^d V &\to V^{\otimes d} \\ v_1 \wedge \cdots \wedge v_d &\mapsto \sum_{\sigma \in \mathfrak{S}_d} (\operatorname{sgn} \sigma) v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)} \end{aligned}$$

($\mathfrak{S}_d$ denotes the symmetric group on d letters), the second map is a certain reordering that we will explain shortly, and the third map is multiplication, i.e., the quotient map $V^{\otimes d} \to \mathrm{Sym}^d V$.

The reordering is best understood in terms of Young diagrams, which we will illustrate with an example.

Example 1.3.1. Consider $\lambda = (3,2)$ so that $\mu = (2,2,1)$. Then $\bigwedge^2 V \otimes \bigwedge^2 V \otimes V$ is spanned by elements of the form $(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5$. We can record element of $V^{\otimes 5}$ by putting vectors into the boxes of $Y(\lambda)$. The order we do this in depends: if we write $V^{\otimes 5} = V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$, then we will think of these as the columns of $Y(\lambda)$. On the other hand, writing $V^{\otimes 5} = V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$, we will instead think of these as the rows of the $Y(\lambda)$. The map then looks as follows:

$$\begin{aligned}(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5 &\mapsto \begin{array}{|c|c|c|}\hline v_1 & v_3 & v_5 \\ \hline v_2 & v_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline v_2 & v_3 & v_5 \\ \hline v_1 & v_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline v_1 & v_4 & v_5 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline v_2 & v_4 & v_5 \\ \hline v_1 & v_3 \\ \cline{1-2}\end{array} \\ &\mapsto v_1v_3v_5 \otimes v_2v_4 - v_2v_3v_5 \otimes v_1v_4 - v_1v_4v_5 \otimes v_2v_3 + v_2v_4v_5 \otimes v_1v_3. \square\end{aligned}$$

Since all of the maps are $\mathbf{GL}(V)$-equivariant, we see that $\mathbf{S}_\lambda(V)$ is a $\mathbf{GL}(V)$-representation. It follows immediately from the definition that $\mathbf{S}_\lambda(V) = 0$ if $\ell(\lambda) > \dim V$ since the corresponding exterior power $\bigwedge^{\mu_1} V$ is 0.

Example 1.3.2. There are two extreme cases that we already know. If $\lambda = (d)$, then the map becomes the quotient map $V^{\otimes d} \to \mathrm{Sym}^d V$ so that $\mathbf{S}_{(d)} V = \mathrm{Sym}^d V$. On the other hand, if $\lambda = (1^d)$, then the map becomes the comultiplication map $\bigwedge^d V \to V^{\otimes d}$, which is injective, so $\mathbf{S}_{(1^d)} V = \bigwedge^d V$. $\square$

Fix a basis $e_1, \dots, e_n$ for V. We would like to find a basis for $\mathbf{S}_\lambda V$. Given a tableau T on $Y(\lambda)$, i.e., a filling of the boxes of $Y(\lambda)$ with the numbers $1, \dots, n$, we get a vector in $\bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V$ by taking

$$(e_{T_{1,1}} \wedge e_{T_{2,1}} \wedge \cdots \wedge e_{T_{\mu_1,1}}) \otimes \cdots \otimes (e_{T_{1,s}} \wedge \cdots \wedge e_{T_{\mu_s,s}});$$

let e_T be its image in $\mathbf{S}_\lambda V$.

We say that T is **semistandard** if $T_{i,j} \le T_{i,j+1}$ and $T_{i,j} < T_{i+1,j}$ for all i, j where that makes sense.

Theorem 1.3.3. *$\{e_T \mid T$ is semistandard$\}$ is a basis for $\mathbf{S}_\lambda V$.*

The proof is elementary, but complicated, so we will omit it.

The e_T are all weight vectors of weight $\mu(T)$ where $\mu(T)_i$ is the number of times that i appears in the tableau T. Note that different tableau can have the same weight. Consider the tableau T where the boxes in row i are filled with i. Then this is a highest weight vector of weight λ since it is the image of a tensor product of highest weight vectors in the exterior powers. In fact, there are no other highest weight vectors (we omit the proof), so we conclude the following theorem:

Theorem 1.3.4. *$\mathbf{S}_\lambda V$ is an irreducible polynomial representation of $\mathbf{GL}(V)$ of highest weight λ.*

We see that the dimension of $\mathbf{S}_\lambda V$ is the number of semistandard Young tableau (SSYT) of shape λ. We give two formulas for this quantity:

Theorem 1.3.5.

$$\dim \mathbf{S}_\lambda(\mathbf{C}^n) = \prod_{1\le i<j\le n} \frac{\lambda_i - \lambda_j + j - i}{j-i}.$$

Given a box (i,j) in the Young diagram of λ, its hook is the set of boxes to the right and below it (including itself). Its **hook length** $h(i,j)$ is the number of boxes in the hook. Below, we list the hook lengths for the partition $(6,3,1)$:

$$\begin{array}{|c|c|c|c|c|c|}\hline 8 & 6 & 5 & 3 & 2 & 1 \\ \hline 4 & 2 & 1 \\ \cline{1-3} 1 \\ \cline{1-1}\end{array}$$

Given a box $(i,j)\in Y(\lambda)$, define its **content** to be $c(i,j) = j-i$.

Theorem 1.3.6 (Hook-content formula)**.**

$$\dim \mathbf{S}_\lambda(\mathbf{C}^n) = \prod_{(i,j)\in Y(\lambda)} \frac{n + c(i,j)}{h(i,j)}.$$

Example 1.3.7. Consider $\lambda = (3,2)$. The first formula gives us (we only have to consider the terms with $i = 1,2$ since otherwise the term is 1):

$$\begin{aligned}\dim \mathbf{S}_{(3,2)}(\mathbf{C}^n) &= (\frac{3-2+1}{2-1}\frac{3+2}{2}\frac{3+3}{3}\cdots\frac{3+(n-1)}{n-1})(\frac{2+1}{1}\frac{2+2}{2}\cdots\frac{2+(n-2)}{n-2}) \\ &= 2\frac{(n+2)!}{4!(n-1)!}\frac{n!}{2(n-2)!} \\ &= \frac{1}{4!}(n+2)(n+1)n^2(n-1).\end{aligned}$$

The second formula gives the last expression directly. □

1.4. Symmetric polynomials and functions.

Lemma 1.4.1. $\mathrm{char}(\rho)(x_1,\dots,x_n)$ *is symmetric, i.e., for any permutation* σ*, we have* $\mathrm{char}(\rho)(x_1,\dots,x_n) = \mathrm{char}(\rho)(x_{\sigma(1)},\dots,x_{\sigma(n)})$.

Proof. Each permutation σ corresponds to a permutation matrix $M(\sigma)$: this is the matrix with a 1 in row $\sigma(i)$ and column i for $i = 1,\dots,n$ and 0's everywhere else. Then

$$M(\sigma)^{-1}\mathrm{diag}(x_1,\dots,x_n)M(\sigma) = \mathrm{diag}(x_{\sigma(1)},\dots,x_{\sigma(n)}).$$

Now use that the trace of a matrix is invariant under conjugation:

$$\begin{aligned}\mathrm{char}(\rho)(x_1,\dots,x_n) &= \mathrm{Tr}(\rho(\mathrm{diag}(x_1,\dots,x_n))) \\ &= \mathrm{Tr}(\rho(M(\sigma))^{-1}\rho(\mathrm{diag}(x_1,\dots,x_n))\rho(M(\sigma))) \\ &= \mathrm{Tr}(\rho(M(\sigma)^{-1}\mathrm{diag}(x_1,\dots,x_n)M(\sigma))) \\ &= \mathrm{Tr}(\rho(\mathrm{diag}(x_{\sigma(1)},\dots,x_{\sigma(n)}))) \\ &= \mathrm{char}(\rho)(x_{\sigma(1)},\dots,x_{\sigma(n)}).\end{aligned}$$ □

By Proposition 1.1.6, we know that every representation becomes polynomial after tensoring with a large enough power of the determinant. This means that every character is of the form $f(x_1,\dots,x_n)/(x_1\cdots x_n)^d$ where f is a symmetric polynomial (i.e., invariant under permutations of the variables).

We denote $\Lambda(n)$ to be the set of symmetric polynomials in n variables $x_1, \dots, x_n$ with integer coefficients. This is a ring under usual addition and multiplication and contains the characters of polynomial representations of $\mathbf{GL}_n(\mathbf{C})$.

We can do something analogous for representations of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$: let $\Lambda(n,m)$ be the set of polynomials in two sets of variables $x_1, \dots, x_n, y_1, \dots, y_m$ with integer coefficients which are symmetric in each set separately.

1.4.1. *Schur polynomials.* The **Schur polynomial** $s_\lambda(x_1, \dots, x_n)$ is the character of $\mathbf{S}_\lambda(\mathbf{C}^n)$. These form a basis for $\Lambda(n)$ (can be proven combinatorially, see Math 202B, or using general results about representation theory, see Math 251AB). We can write it as a sum of $x^{\mu(T)}$ over SSYT T of shape λ where $\mu(T)$ is the weight of T. The *Weyl character formula* can be translated to give a more compact formula which we explain now.

Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a non-negative integer sequence. Define

$$a_\alpha = \det(x_i^{\alpha_j})_{i,j=1}^n = \det \begin{pmatrix} x_1^{\alpha_1} & x_1^{\alpha_2} & \cdots & x_1^{\alpha_n} \\ x_2^{\alpha_1} & x_2^{\alpha_2} & \cdots & x_2^{\alpha_n} \\ \vdots & & & \vdots \\ x_n^{\alpha_1} & x_n^{\alpha_2} & \cdots & x_n^{\alpha_n} \end{pmatrix} = \sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(x^\alpha).$$

Note that a_α is *skew-symmetric*: if we permute a_α by a permutation $\sigma \in \mathfrak{S}_n$, then it changes by $\operatorname{sgn}(\sigma)$. Let $\rho = (n-1, n-2, \dots, 1, 0)$.

Lemma 1.4.2. (a) $\prod_{1 \le i < j \le n}(x_i - x_j)$ *divides every skew-symmetric polynomial in* $x_1, \dots, x_n$.
(b) $a_\rho = \prod_{1 \le i < j \le n}(x_i - x_j)$.

Proof. (a) Let $f(x_1, \dots, x_n)$ be skew-symmetric and let σ be the transposition (i,j). Then $\sigma f = -f$. However, σf and f are the same if we replace x_j by x_i, so this says that specializing x_j to x_i gives 0, i.e., f is divisible by $(x_i - x_j)$. This is true for any i,j, so this proves (a).

(b) a_ρ is divisible by $\prod_{1 \le i < j \le n}(x_i - x_j)$ since it is skew-symmetric. But also note that both are polynomials of degree $1 + 2 + \cdots + (n-1) = \binom{n}{2}$, so they are equal up to some integer multiple. The coefficient of $x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ for both is 1, so they are actually the same. ∎

Define $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$.

Theorem 1.4.3 (Weyl character formula for $\mathbf{GL}_n(\mathbf{C})$)**.** *Given a partition* λ,

$$s_\lambda(x_1, \dots, x_n) = \frac{a_{\lambda+\rho}}{a_\rho} = \frac{\det(x_i^{\lambda_j + n - j})_{i,j=1}^n}{\prod_{1 \le i < j \le n}(x_i - x_j)}.$$

Remark 1.4.4. The Weyl character formula usually takes this form:

$$s_\lambda(x_1, \dots, x_n) = \frac{\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(x^{\lambda+\rho})}{\sum_{\sigma \in \mathfrak{S}_n} \operatorname{sgn}(\sigma)\sigma(x^\rho)}.$$ ∎

1.4.2. *Cauchy identity and some plethysm.* From the way we defined it, it makes sense to apply a Schur functor to any representation W of $\mathbf{GL}_n(\mathbf{C})$: $\mathbf{S}_\lambda(W)$. If we think of this as a representation of $\mathbf{GL}_n(\mathbf{C})$ (rather than $\mathbf{GL}(W)$), then we can get an expression for its character as follows. Let $w_1, \dots, w_N$ be the weights (repeated as necessary) of W. Then

$$\operatorname{char}(\mathbf{S}_\lambda(W))(x_1, \dots, x_n) = s_\lambda(x^{w_1}, \dots, x^{w_N}).$$

We denote this by $s_\lambda \circ \mathrm{char}(W)$. It is then possible to write this as a linear combination of Schur polynomials in the x_i which tells us how the representation decomposes into irreducible representations. Explicitly doing this is hard in general, though in specific examples it can be done with basic linear algebra. More generally, any symmetric polynomial f is a linear combination of Schur polynomials $\sum c_\lambda s_\lambda(x)$, so we can $f \circ \mathrm{char}(W) = \sum_\lambda c_\lambda s_\lambda \circ \mathrm{char}(W)$ for any symmetric polynomial f. Actually, it is possible to extend the definition further and allow $\mathrm{char}(W)$ to be any symmetric polynomial as well, though we won't do that. This operation is called **plethysm**.

Now we'll translate the multiplicity-free actions studied earlier into symmetric polynomial identities. To do that, we will work with power series in a new variable t whose coefficients lie in $\Lambda(n)$. First we need a simple identity. Let $h_d(x) = \mathrm{char}(\mathrm{Sym}^d \mathbf{C}^n)$.

Lemma 1.4.5.
$$\sum_{d\geq 0} h_d(x)t^d = \frac{1}{\prod_{i=1}^n (1-x_i t)}.$$

Proof. The left side is the sum of $x_1^{d_1}\cdots x_n^{d_n} t^{d_1+\cdots+d_n}$ over all choices of non-negative integers $d_1,\dots,d_n$. Using the geometric series, the right side is the product
$$\prod_{i=1}^n \Big(\sum_{d_i\geq 0} x_i^{d_i} t^{d_i}\Big),$$
which is the same when expanded out. □

Theorem 1.4.6 (Cauchy identity)**.**
$$\prod_{i=1}^n \prod_{j=1}^m (1-x_i y_j t)^{-1} = \sum_\lambda s_\lambda(x_1,\dots,x_n) s_\lambda(y_1,\dots,y_m) t^{|\lambda|}$$
where the sum is over all integer partitions with $\lambda_1 \geq \cdots \geq \lambda_{\min(n,m)} \geq 0$.

Proof. By Corollary 1.2.11, we have
$$\sum_{d\geq 0} h_d(x_1y_1,\dots,x_ny_m)t^d = \sum_\lambda s_\lambda(x_1,\dots,x_n)s_\lambda(y_1,\dots,y_m)t^{|\lambda|}.$$
Use the previous lemma to replace the left side. □

Similarly, we get the following two identities using Corollary 1.2.15 and Corollary 1.2.19:

Theorem 1.4.7.
$$\prod_{1\leq i\leq j\leq n} (1-x_ix_jt)^{-1} = \sum_\lambda s_{2\lambda}(x_1,\dots,x_n)t^{|\lambda|},$$
$$\prod_{1\leq i< j\leq n} (1-x_ix_jt)^{-1} = \sum_\lambda s_{(2\lambda)^\dagger}(x_1,\dots,x_n)t^{|\lambda|}.$$

1.4.3. *Symmetric functions.* We have maps $\pi_n \colon \Lambda(n+1) \to \Lambda(n)$ obtained by $f(x_1,\dots,x_n,x_{n+1}) \mapsto f(x_1,\dots,x_n,0)$. This gives us a way to compare representations of $\mathbf{GL}_{n+1}(\mathbf{C})$ with $\mathbf{GL}_n(\mathbf{C})$. It follows from our interpretation in terms of SSYT that $\pi_n(s_\lambda(x_1,\dots,x_{n+1})) = s_\lambda(x_1,\dots,x_n)$.

We define Λ to be the *graded inverse limit* of the system $\{\pi_n \colon \Lambda(n+1) \to \Lambda(n)\}$. Explicitly, a degree d element $f \in \Lambda$ is a sequence of degree d elements $f_n \in \Lambda(n)$ such that $\pi_n(f_{n+1}) = f_n$ for all n. A general element of Λ is a finite sum of degree d elements. Concretely, we can think of elements $f \in \Lambda$ as bounded-degree power series in $x_1, x_2, \dots$ which are invariant under all permutations of the variables. The connection is that $f_n = f(x_1, \dots, x_n, 0, 0, \dots)$. Elements of Λ are called **symmetric functions**.

An example is given by the sequence of Schur polynomials $s_\lambda(x_1, \dots, x_n)$ (if $\ell(\lambda) > n$, then this is defined to be 0), and the limit symmetric function is the **Schur function** s_λ. They form a basis for Λ.

Note that Λ is also a ring under the usual addition and multiplication operations and that the specialization maps $\Lambda \to \Lambda(n)$ obtained by $x_{n+1} = x_{n+2} = \cdots = 0$ are ring homomorphisms. Explicitly, this means that if we do computations, such as multiplication or plethysm, we can do it in the ring Λ and then we automatically get answers in $\Lambda(n)$ for all n. Heuristically: "the representation theory of polynomial representations of $\mathbf{GL}_n(\mathbf{C})$ exhibits *stability* with respect to n". This will not hold for the symplectic and orthogonal groups in general, though will hold if we require $n \gg 0$.

1.4.4. *Littlewood–Richardson coefficients.* Since the Schur functions s_λ form a basis for Λ, there exist integers $c^\lambda_{\mu,\nu}$ such that $s_\mu s_\nu = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda$. These are the **Littlewood–Richardson coefficients**. There are various combinatorial ways to compute them, but we will not go into the general case. But here are two properties:

- If $c^\lambda_{\mu,\nu} \neq 0$, then $\mu \subseteq \lambda$ and $\nu \subseteq \lambda$.
- $c^\lambda_{\mu,\nu} = c^{\lambda^\dagger}_{\mu^\dagger,\nu^\dagger}$

By specializing to n variables, we see that

$$s_\mu(x_1, \dots, x_n) s_\nu(x_1, \dots, x_n) = \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda(x_1, \dots, x_n)$$

with the understanding that $s_\lambda(x_1, \dots, x_n) = 0$ if $\ell(\lambda) > n$. The left side is the character of the tensor product $\mathbf{S}_\mu(\mathbf{C}^n) \otimes \mathbf{S}_\nu(\mathbf{C}^n)$, and since the Schur polynomials form a basis for $\Lambda(n)$, the right side describes the unique way to decompose this tensor product into irreducible representations, so in particular, $c^\lambda_{\mu,\nu} \geq 0$. This decomposition is basically insensitive to the value of n, and this is one instance of stabilization mentioned above.

There are two important special cases of the Littlewood–Richardson coefficients which are worth mentioning. The first is when $\nu = (k)$, i.e., s_ν is the character of the kth symmetric power and the second is when $\nu = (1^k)$, i.e., when s_ν is the character of the kth exterior power. Then we have the following rules:

Theorem 1.4.8 (Pieri rule)**.**

$$s_\mu s_k = \sum_{\substack{\lambda \text{ s.t.} \\ |\lambda| = |\mu| + k, \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots}} s_\lambda, \qquad s_\mu s_{(1^k)} = \sum_{\substack{\lambda \text{ s.t.} \\ |\lambda| = |\mu| + k, \\ \lambda_1^\dagger \geq \mu_1^\dagger \geq \lambda_2^\dagger \geq \mu_2^\dagger \geq \cdots}} s_\lambda.$$

We can give another interpretation as follows. Say that λ/μ is a **horizontal strip** if no column in the skew Young diagram of λ/ν contains 2 or more boxes. Similarly, say that λ/μ is a **vertical strip** if no row in the skew Young diagram of λ/μ contains 2 or more boxes.

Theorem 1.4.9 (Pieri rule)**.** • *If $\nu = (k)$, then*

$$c^\lambda_{\mu,(k)} = \begin{cases} 1 & \text{if } |\lambda| = |\mu| + k \text{ and } \lambda/\mu \text{ is a horizontal strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_\mu s_k = \sum_\lambda s_\lambda$$

where the sum is over all λ such that λ/μ is a horizontal strip of size k.

- *If $\nu = (1^k)$, then*

$$c^\lambda_{\mu,(1^k)} = \begin{cases} 1 & \text{if } |\lambda| = |\mu| + k \text{ and } \lambda/\mu \text{ is a vertical strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_\mu s_{1^k} = \sum_\lambda s_\lambda$$

where the sum is over all λ such that λ/μ is a vertical strip of size k.

2. Symplectic groups

2.1. **Definitions and basic properties.**

Recall that we defined an action of $\mathbf{GL}_m(\mathbf{C})$ on $\bigwedge^2 \mathbf{C}^m$, the space of $m \times m$ skew-symmetric matrices, by

$$g \cdot u = (g^{-1})^T u g^{-1}.$$

We will assume that $m = 2n$ is even. The **symplectic group** $\mathbf{Sp}_{2n}(\mathbf{C})$ is the stabilizer of a full rank matrix. Note that any two full rank matrices are in the same orbit, so any two choices lead to conjugate subgroups of $\mathbf{GL}_{2n}(\mathbf{C})$, and in particular are isomorphic.

For concreteness, let I' be the $n \times n$ matrix which is 1 on the antidiagonal, i.e., $I'_{i,j} = 1$ if $j = n + 1 - i$ and 0 otherwise, and we take the following skew-symmetric matrix:

$$\Omega = \begin{bmatrix} 0 & I' \\ -I' & 0 \end{bmatrix}.$$

In symbols,

$$\mathbf{Sp}_{2n}(\mathbf{C}) = \{g \mid g^T \Omega g = \Omega\} = \{g \in \mathbf{GL}_{2n}(\mathbf{C}) \mid \Omega^{-1}(g^T)^{-1}\Omega = g\}.$$

We take m even so that this is a semisimple group (otherwise it is not). Note that either description shows that $\mathbf{Sp}_{2n}(\mathbf{C})$ is the zero set of polynomial functions on the space of $2n \times 2n$ matrices, and hence is an affine variety (since we can realize $\mathbf{GL}$ as an affine variety).

There is a definition of (Krull) dimension for algebraic varieties which is too complicated to get into now, but we will note a few properties:

1. $\dim \mathbf{GL}_n(\mathbf{C}) = n^2$
2. If X is a vector space, then $\dim X$ is the usual dimension of X as a vector space.
3. If U is a nonempty open subset of an irreducible variety, then $\dim U = \dim X$.
4. If U is an orbit of G and H is the stabilizer of a point in U, then $\dim G - \dim H = \dim U$.
5. $\dim(X \times Y) = \dim X + \dim Y$

Combining these properties, we see that

$$\dim \mathbf{Sp}_{2n}(\mathbf{C}) = \dim \mathbf{GL}_{2n}(\mathbf{C}) - \dim \bigwedge^2 \mathbf{C}^{2n} = (2n)^2 - \binom{2n}{2} = n(2n+1).$$

The matrix Ω defines a bilinear form on $\mathbf{C}^{2n}$ as follows: given $v, w \in \mathbf{C}^{2n}$, we define

$$\omega(v, w) = v^T \Omega w.$$

Since $\Omega^T = -\Omega$, we see that $\omega(v,w) = -\omega(w,v)$, so it is skew-symmetric. Furthermore, since $\operatorname{rank}\Omega = 2n$, ω is **non-degenerate**: for all v, there exists w such that $\omega(v,w) \neq 0$. A **symplectic form** is, by definition, a skew-symmetric non-degenerate bilinear form. With respect to the standard basis $e_1, \dots, e_{2n}$, we have

$$\omega(e_i, e_{2n+1-j}) = \delta_{i,j} \text{ for } 1 \le i \le n.$$

We will call any basis with this property a **symplectic basis**.

A representation is a homomorphism $\rho\colon \mathbf{Sp}_{2n}(\mathbf{C}) \to \mathbf{GL}(W)$ for some vector space W whose entries are rational functions after picking a basis for W. Much of the theory from $\mathbf{GL}_{2n}(\mathbf{C})$ carries over. We repeat it here with the updated terms.

Given a symplectic basis, we let the Borel subgroup B be the group of upper triangular matrices in $\mathbf{Sp}_{2n}(\mathbf{C})$ and the maximal torus T be the group of diagonal matrices. The latter are diagonal matrices with entries $x_1, x_2, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$.

If $\rho\colon \mathbf{Sp}_{2n}(\mathbf{C}) \to \mathbf{GL}(W)$ is a representation, then there is a basis $w_1, \dots, w_r$ for W such that each w_i is an eigenvector of $\rho(t)$ for all $t \in T$; if the entries of t are $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$, then the eigenvalues are of the form $x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$ for some $\mu_{i,1}, \dots, \mu_{i,n} \in \mathbf{Z}$. Any vector w which is an eigenvector for all $\rho(t)$ is called a **weight vector** and $(\mu_1, \dots, \mu_n)$ is its **weight** if $\rho(t)(w) = x_1^{\mu_1} \cdots x_n^{\mu_n} w$ for all $x_1, \dots, x_n$. We also write $\mu(t)$ for $x_1^{\mu_1} \cdots x_n^{\mu_n}$.

The **character** of ρ is defined to be the function $\operatorname{char}(\rho)(x_1, \dots, x_n) = \operatorname{Tr}\rho(t)$ where t is the diagonal matrix with entries $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$ and Tr denotes trace. Alternatively, $\operatorname{char}(\rho)(x_1, \dots, x_n) = \sum_{i=1}^r x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$.

A nonzero vector $w \in W$ is a **highest weight vector** if $b \cdot w$ is a scalar multiple of w for all $b \in B$. In particular, it is an eigenvector for all $\rho(t)$ for $t \in T$, so it is also a weight vector. If μ is the weight of this vector, we write $\mu(b)$ for the scalar multiple i.e., $b \cdot w = \mu(b)w$.

We can define direct sums, tensor products, and duals in the same way and they transform characters in the same way.

Theorem 2.1.1. *(1) Every finite-dimensional representation of $\mathbf{Sp}_{2n}(\mathbf{C})$ is semisimple.*

(2) Two representations of $\mathbf{Sp}_{2n}(\mathbf{C})$ are isomorphic if and only if they have the same character.

(3) (Schur's lemma) There are no nonzero homomorphisms between non-isomorphic simple representations. Any homomorphism from a simple representation to itself must be a scalar multiple of the identity.

(4) Every finite-dimensional representation contains a highest weight vector.

(5) Any two highest weight vectors (for a particular choice of Borel subgroup) in an irreducible representation are scalar multiples of each other.

(6) The weight $(\mu_1, \dots, \mu_n)$ of a highest weight vector satisfies $\mu_1 \ge \cdots \ge \mu_n \ge 0$.

(7) For every $(\mu_1, \dots, \mu_n) \in \mathbf{Z}^n$ satisfying $\mu_1 \ge \cdots \ge \mu_n \ge 0$, there is an irreducible representation whose highest weight vector has that weight.

We will denote the irreducible representation whose highest weight is μ by $\mathbf{S}_{[\mu]}(\mathbf{C}^{2n})$.

Example 2.1.2. (1) Take ρ to be the natural inclusion $\mathbf{Sp}_{2n}(\mathbf{C}) \to \mathbf{GL}_{2n}(\mathbf{C})$, so $W = \mathbf{C}^{2n}$. The standard basis vectors $e_1, \dots, e_{2n}$ are weight vectors where for $i = 1, \dots, n$, the weight of e_i is $(0, \dots, 1, \dots, 0)$ (with a 1 in position i) and the weight of e_{2n+1-i} is $(0, \dots, -1, \dots, 0)$ (with a -1 in position i). The highest weight vector is e_1 and the character is $\sum_{i=1}^n (x_i + x_i^{-1})$. Note that this implies that W is isomorphic to its dual representation since its character is invariant under $x_i \mapsto x_i^{-1}$.

(2) In general, if $\rho\colon \mathbf{GL}_{2n}(\mathbf{C}) \to \mathbf{GL}(W)$ is a rational representation, then we get a rational representation by restricting to $\mathbf{Sp}_{2n}(\mathbf{C})$. We denote this representation by $\rho|_{\mathbf{Sp}_{2n}(\mathbf{C})}$. Then we have

$$\mathrm{char}(\rho|_{\mathbf{Sp}_{2n}(\mathbf{C})})(x_1,\dots,x_n) = \mathrm{char}(\rho)(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}).$$

(3) Consider the second exterior power $\bigwedge^2 \mathbf{C}^{2n}$ with $n \ge 2$. Its character as a $\mathbf{GL}_{2n}(\mathbf{C})$ representation (using y variables) is $\sum_{1\le i<j\le 2n} y_iy_j$. So the character of its restriction to $\mathbf{Sp}_{2n}(\mathbf{C})$ is $\sum_{1\le i<j\le n}(x_ix_j + x_i^{-1}x_j^{-1}) + \sum_{i\ne j} x_ix_j^{-1} + n$. By definition of the symplectic group, the line spanned by Ω is a trivial subrepresentation of $\bigwedge^2 \mathbf{C}^{2n}$. Let $\bigwedge^2_0 \mathbf{C}^{2n}$ denote a complementary subrepresentation. In fact, this is irreducible: the only weights that appear in it that potentially be highest weights are $(1,1,0,\dots,0)$ and $(0,0,\dots,0)$. We leave it as an exercise to show that there are no further trivial subrepresentations. $\square$

A general question that we will address: given an irreducible $\mathbf{GL}_{2n}(\mathbf{C})$ representation $\mathbf{S}_\lambda(\mathbf{C}^{2n})$, how do we determine the decomposition of its restriction to $\mathbf{Sp}_{2n}(\mathbf{C})$ as a direct sum of irreducible representations? This is known as a **branching problem**. We will see that the answer is uniform in n as long as $n \ge \ell(\lambda)$, but otherwise there are complications (which we will also discuss).

The following fact will be used and we sketch a proof in the homework:

Proposition 2.1.3. $\mathbf{GL}_n(\mathbf{C})$ *and* $\mathbf{Sp}_{2n}(\mathbf{C})$ *are connected in the Zariski topology.*

2.2. **Multiplicity-free action.** We will study a multiplicity-free action of $\mathbf{GL}_n(\mathbf{C})\times\mathbf{Sp}_{2n}(\mathbf{C})$. This representation will be very helpful for understanding the branching problem. The results about representations carry over to this product group so we won't restate them.

Given a symplectic form ω on $\mathbf{C}^{2n}$, a subspace $V \subset \mathbf{C}^{2n}$ is **isotropic** if $\omega(x,y)=0$ for all $x,y \in V$. A few basic properties (which we leave to exercises):

(1) Every 1-dimensional subspace is isotropic.
(2) If V is isotropic, then $\dim V \le n$. In this case, V is called **Lagrangian**.
(3) Given 2 isotropic subspaces V_1, V_2 with $\dim V_1 = \dim V_2$, there exists $g \in \mathbf{Sp}_{2n}(\mathbf{C})$ such that $gV_1 = V_2$.
(4) Every isotropic subspace is contained in a Lagrangian subspace.

Now consider the space $U = (\mathbf{C}^{2n}\otimes\mathbf{C}^n)^*$ which we identify with $2n\times n$ matrices. We define an action of $\mathbf{GL}_n(\mathbf{C})\times\mathbf{Sp}_{2n}(\mathbf{C})$ by

$$(g,h)\cdot u = (h^{-1})^T u g^{-1}.$$

Define

$$X = \{u \in U \mid u^T\Omega u = 0\}.$$

Since $u^T\Omega u$ is a skew-symmetric matrix whose entries are polynomial functions of the entries of u, we see that X is an affine variety which is the solution set of $\binom{n}{2}$ polynomials. Alternatively, X is the subset of maps $u\colon \mathbf{C}^n \to \mathbf{C}^{2n}$ such that $u(\mathbf{C}^n)$ is an isotropic subspace. Then X is closed under the action of $\mathbf{GL}_n(\mathbf{C})\times\mathbf{Sp}_{2n}(\mathbf{C})$: if $u\in X$, then using that $(h^{-1})^T = \Omega h \Omega^T$, we have

$$(g^{-1})^Tu^T\Omega h^T\Omega^T\Omega\Omega h\Omega^T ug^{-1} = (g^{-1})^Tu^T\Omega ug^{-1} = (g^{-1})^T0g^{-1} = 0.$$

Proposition 2.2.1. X *is irreducible.*

Proof. Let X° be the subset of matrices of full rank n, i.e., where the map $u\colon \mathbf{C}^n \to \mathbf{C}^{2n}$ is injective. Then $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{Sp}_{2n}(\mathbf{C})$ acts transitively on X°: the image is a Lagrangian subspace and $\mathbf{Sp}_{2n}(\mathbf{C})$ acts transitively on those, while for a fixed Lagrangian L, the set of linear isomorphisms $\mathbf{C}^n \to L$ is a single orbit under pre-composition by $\mathbf{GL}_n(\mathbf{C})$.

Picking any point in $x \in X^\circ$, we get a surjective map $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{Sp}_{2n}(\mathbf{C}) \to X^\circ$ via $(g,h) \mapsto (g,h)\cdot x$. Since the target is connected, it is also irreducible (exercise). This implies that the target is also irreducible (the image of an irreducible space is always irreducible).

Next, we show that X° is dense in X which finishes the proof (if $X = X_1 \cup X_2$ both closed, then $X^\circ = X_i \cap X^\circ$ for some i by irreducibility, i.e., X° is contained in one of the X_i, but denseness implies $X_i = X$). Let $Y \subset X$ be a closed subset containing X° and assume $Y \neq X$. Pick $u \in X \setminus Y$, i.e., $\operatorname{rank} u = r < n$. We can find a basis $v_1, \dots, v_n$ for $\mathbf{C}^n$ so that $u(v_1), \dots, u(v_r)$ are linearly independent and $u(v_{r+1}) = \cdots = u(v_n) = 0$. By the properties above, we can find vectors $w_{r+1}, \dots, w_n \in \mathbf{C}^{2n}$ so that $u(v_1), \dots, u(v_r), w_{r+1}, \dots, w_n$ spans an isotropic subspace of dimension n.

Now define $\varphi\colon \mathbf{C} \to X$ by letting $\varphi(t)$ be the function $\varphi(t)(v_i) = u(v_i)$ for $i = 1, \dots, r$ and $\varphi(t)(v_i) = tw_i$ for $i = r+1, \dots, n$. Then φ is continuous and $\varphi(0) = u$ while $\varphi(t) \in X^\circ$ for $t \neq 0$. In particular, $\varphi^{-1}(Y) = \mathbf{C} \setminus 0$ is closed, but then that implies $\mathbf{C} = \varphi^{-1}(Y) \cup 0$ is reducible, a contradiction. Hence u does not exist, so $Y = X$, which means X° is dense. $\square$

Let J be the $2n \times n$ matrix with $J_{i,i} = 1$ for $i = 1, \dots, n$ and 0's elsewhere. Let $B \subset \mathbf{GL}_n(\mathbf{C})$ be the subgroup of upper-triangular matrices, and similarly, let $B' \subset \mathbf{Sp}_{2n}(\mathbf{C})$ be the subgroup of upper-triangular matrices.

Let $f_i \in \mathbf{C}[X]$ be the function which takes the determinant of the upper-left $i \times i$ submatrix.

Proposition 2.2.2. *f_i is a highest weight vector with weight $(1,\dots,1,0,\dots,0), (1,\dots,1,0,\dots,0)$ (the number of 1's in each vector is i).*

Proof. Pick upper-triangular matrices $g \in \mathbf{GL}_n(\mathbf{C})$ and $h \in \mathbf{Sp}_{2n}(\mathbf{C})$ and write $g = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$ and $h = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix}$ where x_1, x_2 are $i \times i$. Then $(g,h) \cdot f_i$ is the determinant of the upper-left $i \times i$ submatrix of $h^T ug$, which is $\det(x_1)\det(x_2) f_i$. In particular, it is a highest weight vector. If g, h are diagonal, then $\det(x_1)\det(x_2)$ is the product of the first i entries of g and h, so we get the weight also. $\square$

Lemma 2.2.3. *The $B \times B'$ orbit in X containing J is open and dense.*

Proof. We claim that the orbit is precisely the set of matrices A such that $f_i(A) \neq 0$ for $i = 1, \dots, n$. It is easy to see that being in the orbit implies the condition on submatrices, so we just prove the reverse direction. Before proving the claim, note that this shows that the orbit is $\bigcap_{i=1}^n (U \setminus Z(f_i))$, which is open (denseness follows from Propositions 1.2.4 and 1.2.5).

The action of B is by column operations (we can add any multiple of a column to one that appears to the right of it and also scale columns). Since $f_1(A) \neq 0$, we can use column operations to assume that the first row is $(1, 0, \dots, 0)$. Note this doesn't change the fact that the f_i are nonzero. Using $f_2 \neq 0$, we see the $(2,2)$ entry is nonzero and then assume the second row is $(x, 1, 0, \dots, 0)$ (for some $x \in \mathbf{C}$). Continuing in this way, we can use an element of B to turn the top $n \times n$ matrix into a lower-triangular matrix with 1's on the diagonal.

Similarly, B' acts by row operations (we can add any multiple of a row to one that appears below it and also scale) which preserve the symplectic form. An arbitrary invertible linear transformation g of the basis vectors $e_1, \dots, e_n$ can be extended to the symplectic group element $\begin{bmatrix} g & 0 \\ 0 & (I')^{-1}(g^T)^{-1}I' \end{bmatrix}$, so we can also do row operations to assume the top $n \times n$ matrix of A is the identity.

So far, we know that our matrix is in the $B \times B'$ orbit of $\begin{bmatrix} I_n \\ h \end{bmatrix}$. Since this belongs to X, we get $-h^T I' + I'h = 0$. Multiplying by I' on the left and right we get $-I'h^T + hI' = 0$ and implies that $H = \begin{bmatrix} I & -h^T \\ 0 & I \end{bmatrix} \in \mathbf{Sp}_{2n}(\mathbf{C})$. Finally, $(1, H^{-1}) \cdot \begin{bmatrix} I_n \\ h \end{bmatrix} = J$. ☐

Lemma 2.2.4. *If (λ, λ') is the weight of a highest weight vector in $\mathbf{C}[X]$, then $\lambda_i = \lambda'_i$ for $1 \le i \le n$.*

Proof. The stabilizer of J contains pairs of diagonal matrices (g, h) where the entries of h are $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$ and the entries of g are $x_1^{-1}, \dots, x_n^{-1}$. By Theorem 1.2.6, we see that $x_1^{\lambda_1 - \lambda'_1} \cdots x_n^{\lambda_n - \lambda'_n} = 1$ for all $x_1, \dots, x_m$. This forces all of the exponents to be 0. ☐

Finally, each pair (λ, λ') as above is the highest weight for some irreducible representation in $\mathbf{C}[X]$ by Proposition 1.2.7 since every partition λ is a sum of vectors of the form $(1, 1, \dots, 1, 0, \dots, 0)$.

Corollary 2.2.5 (Symplectic Cauchy identity)**.** *We have an isomorphism of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{Sp}_{2n}(\mathbf{C})$ representations*

$$\mathbf{C}[X] \cong \bigoplus_{\lambda} \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$$

where the sum is over all integer partitions $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$.

2.3. **Branching problem.** We can use X from the previous section to study the branching problem from $\mathbf{GL}_{2n}(\mathbf{C})$ to $\mathbf{Sp}_{2n}(\mathbf{C})$. Our approach uses some theorems from commutative algebra, but I will reformulate them in a way that is usable for us.

First, we revisit the equations defining X. We said $X = \{u \mid u^T \Omega u = 0\}$. Since $\Omega = -\Omega$, $u^T \Omega u$ is a skew-symmetric $n \times n$ matrix. So taking its upper-triangular entries, we get $\binom{n}{2}$ equations whose common solution set is X.

Proposition 2.3.1. $\dim X = (3n^2 + n)/2 = \dim U - \binom{n}{2}$.

Proof. We first describe the stabilizer subgroup of J in $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{Sp}_{2n}(\mathbf{C})$. Since it's closed under inverses, we know that $(g, h) \cdot J = J$ implies that $h^T J g = J$. Write $h = \begin{bmatrix} h_1 & h_2 \\ h_3 & h_4 \end{bmatrix}$ which are $n \times n$. Then this condition implies that $h_1 = (g^{-1})^T$ and $h_2 = 0$. The fact that $h^T \Omega h = \Omega$ then implies that $h_4 = I' g I'$ and $-h_3^T I' g^{-1} + (g^{-1})^T I' h_3 = 0$. Multiply this last condition by $I' g^T$ on the left and g on the right to get $I'(h_3 g)^T I' - h_3 g$. This condition says that $h_3 g$ is symmetric with respect to the antidiagonal, so in particular, the space of such choices is just a translate of a vector space of dimension $\binom{n+1}{2}$. The space of choices for g has dimension n^2, so the stabilizer subgroup has dimension $n^2 + \binom{n+1}{2}$.

This means the orbit of J has dimension

$$\dim(\mathbf{GL}_n(\mathbf{C})) + \dim \mathbf{Sp}_{2n}(\mathbf{C}) - (n^2 + \binom{n+1}{2}) = \frac{3n^2 + n}{2}.$$

Since the orbit is dense in X, this is also $\dim X$. □

Krull's principal ideal theorem implies that given affine varieties $X \subset Y$ such that X is the solution set of r polynomials in $\mathbf{C}[Y]$, we have $\dim X \geq \dim Y - r$. When equality is achieved, as in our example, the set of r polynomials is called a **system of parameters**. If furthermore Y is a vector space, then every system of parameters is a **regular sequence** (we will move the definitions and some properties to homework). Here is an important consequence for us:

Theorem 2.3.2. *Let U be a vector space and let $f_1, \dots, f_r \in \mathbf{C}[U]$ be a regular sequence of homogeneous polynomials. Pick a basis $\{g'_i\}$ for $\mathbf{C}[U]/(f_1, \dots, f_r)$ and let $g_i \in \mathbf{C}[U]$ be a coset representative for g'_i. Then the map given by*

$$\begin{aligned} \mathbf{C}[U]/(f_1, \dots, f_r) \otimes \mathbf{C}[y_1, \dots, y_r] &\to \mathbf{C}[U] \\ g'_i \otimes p(y) &\mapsto g_i p(f_1, \dots, f_r) \end{aligned}$$

is an isomorphism of vector spaces.

This will be more useful for us with representations.

Corollary 2.3.3. *Let U be a representation and let $f_1, \dots, f_r \in \mathbf{C}[U]$ be a regular sequence of homogeneous polynomials such that their span is a subrepresentation V. Then we have an isomorphism of representations*

$$\mathbf{C}[U]/(f_1, \dots, f_r) \otimes \mathrm{Sym}(V) \cong \mathbf{C}[U].$$

Let's unpackage this for our example. We take U as in this section and $f_1, \dots, f_r$ are the entries of the skew-symmetric matrix $u^T \Omega u$ with $r = \binom{n}{2}$. From the calculation in the previous section, we get

$$((g,h) \cdot u)^T \Omega ((g,h) \cdot u) = (g^{-1})^T (u^T \Omega u) g^{-1}$$

so that we see that $f_1, \dots, f_r$ span a subrepresentation V of $\mathbf{C}[U]$ where $V \cong \bigwedge^2 \mathbf{C}^n$ (with the $\mathbf{Sp}_{2n}(\mathbf{C})$ factor acting trivially). This implies that we have an isomorphism of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{Sp}_{2n}(\mathbf{C})$ representations

$$\mathbf{C}[U]/(f_1, \dots, f_r) \otimes \mathrm{Sym}(\bigwedge^2 \mathbf{C}^n) \cong \mathbf{C}[U].$$

We want to identify $\mathbf{C}[U]/(f_1, \dots, f_r)$ with $\mathbf{C}[X]$. The potential issue is this: there might exist polynomials that are identically 0 on X but which are not generated by $f_1, \dots, f_r$. Fortunately this is not the case:

Proposition 2.3.4. *Any polynomial that is identically* 0 *on X is generated by the entries of $u^T \Omega u$. In particular, $\mathbf{C}[X] = \mathbf{C}[U]/(f_1, \dots, f_r)$.*

This follows (though we leave it as an exercise) from the following theorem (which we will not prove, but a more general statement appears as [E, Exercise 11.10]):

Theorem 2.3.5 (Serre's criterion, special case)**.** *Let X be an affine variety in a vector space U such that $X = Z(f_1, \dots, f_r)$ for a regular sequence $f_1, \dots, f_r$. Let $x_1, \dots, x_N$ be a basis for U^* and consider the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})$. If X is irreducible and there exists a point $\alpha \in X$ such that $(\frac{\partial f_i}{\partial x_j}(\alpha))$ has maximal rank r, then $\mathbf{C}[X] = \mathbf{C}[U]/(f_1, \dots, f_r)$.*

(This is also true if X is not irreducible, provided that we can find a point α with the above property lying on each irreducible component of X, i.e., the maximal irreducible subsets.)

Remark 2.3.6 (Weyl's construction)**.** $\mathbf{C}[X]$ is a quotient of $\mathbf{C}[U]$ and the kernel is generated by $\bigwedge^2(\mathbf{C}^n)$, so in particular the kernel is a quotient $\mathbf{C}[U] \otimes \bigwedge^2 \mathbf{C}^n$, i.e., we have an exact sequence

$$\mathbf{C}[U] \otimes \bigwedge^2 \mathbf{C}^n \to \mathbf{C}[U] \to \mathbf{C}[X] \to 0.$$

We can consider the $\mathbf{S}_\lambda(\mathbf{C}^n)$ components of this sequence to get

$$\bigoplus_\mu \mathbf{S}_\mu(\mathbf{C}^{2n})^{\oplus c^\lambda_{\mu,(1,1)}} \to \mathbf{S}_\lambda(\mathbf{C}^{2n}) \to \mathbf{S}_{[\lambda]}(\mathbf{C}^{2n}) \to 0.$$

The first term simplifies a bit: we just sum over μ such that $|\lambda| = |\mu| + 2$ and λ/μ is a vertical strip. This gives us a recipe for constructing $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$ using Schur functors: given μ and λ as above, we have a map

$$\mathbf{S}_\mu(\mathbf{C}^{2n}) \otimes \bigwedge^2(\mathbf{C}^{2n}) \to \mathbf{S}_\lambda(\mathbf{C}^{2n})$$

by Pieri's rule. We can identify $\mathbf{S}_\mu(\mathbf{C}^{2n})$ as the subspace $\mathbf{S}_\mu(\mathbf{C}^{2n}) \otimes \langle \Omega \rangle$ of the source, and then the image of this subspace is a $\mathbf{Sp}_{2n}(\mathbf{C})$-subrepresentation of $\mathbf{S}_\lambda(\mathbf{C}^{2n})$. If we quotient by all of these subrepresentations, then we get the irreducible $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$. This is also known as the space of "traceless tensors". $\square$

In our situation, we get

$$\mathbf{C}[X] \otimes \mathrm{Sym}(\bigwedge^2 \mathbf{C}^n) \cong \mathrm{Sym}(\mathbf{C}^n \otimes \mathbf{C}^{2n}). \tag{2.3.7}$$

Using all of the various multiplicity-free representations we've studied this becomes the following identity:

$$\left(\bigoplus_\mu \mathbf{S}_\mu(\mathbf{C}^n) \otimes \mathbf{S}_{[\mu]}(\mathbf{C}^{2n})\right) \otimes \left(\bigoplus_\nu \mathbf{S}_{(2\nu)^\dagger}(\mathbf{C}^n)\right) \cong \bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_\lambda(\mathbf{C}^{2n}).$$

(in the last $\mathbf{S}_\lambda(\mathbf{C}^{2n})$, this is the restriction of the Schur functor to $\mathbf{Sp}_{2n}(\mathbf{C})$). The left hand side can be combined using Littlewood–Richardson coefficients:

$$\bigoplus_{\mu,\nu,\lambda} (\mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{[\mu]}(\mathbf{C}^{2n}))^{\oplus c^\lambda_{\mu,(2\nu)^\dagger}}.$$

Comparing these two expressions we see how Schur functors decompose into irreducible representations:

Theorem 2.3.8 (Stable branching rule from $\mathbf{GL}$ to $\mathbf{Sp}$)**.** *Let λ be a partition with $\ell(\lambda) \le n$. Then we have an isomorphism of $\mathbf{Sp}_{2n}(\mathbf{C})$-representations*

$$\mathbf{S}_\lambda(\mathbf{C}^{2n}) \cong \bigoplus_\mu \mathbf{S}_{[\mu]}(\mathbf{C}^{2n})^{\oplus m_{\lambda,\mu}}, \qquad m_{\lambda,\mu} = \sum_\nu c^\lambda_{\mu,(2\nu)^\dagger}.$$

The "stable" in the name refers to the condition $\ell(\lambda) \le n$. Note that the answer does not depend on n as long as this condition is satisfied. We can make sense of this question in general, but the method above does not give an answer. We call the case $\ell(\lambda) > n$ the unstable case.

Example 2.3.9. When $\lambda = \mu$, we have $m_{\lambda,\lambda} = 1$ since the only valid choice for ν is the empty partition corresponding to the trivial representation, and $c^{\lambda}_{\lambda,\varnothing} = 1$. The highest weight vector for $\mathbf{GL}_{2n}(\mathbf{C})$ in $\mathbf{S}_\lambda(\mathbf{C}^{2n})$ is also a highest weight vector of weight λ for the subgroup $\mathbf{Sp}_{2n}(\mathbf{C})$. ◻

Example 2.3.10. If $\lambda = (1^d)$, then $c^{\lambda}_{\mu,(2\nu)^\dagger} > 0$ implies that both μ and $(2\nu)^\dagger$ are contained in (1^d), i.e., $Y(\mu)$ is a column of size $\le d$ and $(2\nu)^\dagger$ is an even length column of size $\le d$. So if $d \le n$, we see that

$$\bigwedge^d(\mathbf{C}^{2n}) \cong \mathbf{S}_{[1^d]}(\mathbf{C}^{2n}) \oplus \mathbf{S}_{[1^{d-2}]}(\mathbf{C}^{2n}) \oplus \mathbf{S}_{[1^{d-4}]}(\mathbf{C}^{2n}) \oplus \cdots .$$

To see that the unstable case is different, consider $d = 2n$. Then $\bigwedge^{2n}(\mathbf{C}^{2n})$ is the determinant representation, and this is trivial on $\mathbf{Sp}_{2n}(\mathbf{C})$, so $\bigwedge^{2n}(\mathbf{C}^{2n}) \cong \mathbf{S}_{[0]}(\mathbf{C}^{2n})$. ◻

2.4. **Characters of the symplectic group.** Given a representation $\rho\colon \mathbf{Sp}_{2n}(\mathbf{C}) \to \mathbf{GL}(W)$, its character $\operatorname{char}(\rho)(x_1,\dots,x_n)$ is the trace of the diagonal matrix with entries $x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}$. As with $\mathbf{GL}$-representations, we can show this is symmetric under the group $\mathfrak{S}_n$ permuting variables, but actually we can say more.

Let H_n be the group of permutations σ of $\{x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}\}$ such that $\sigma(1/x_i) = 1/\sigma(x_i)$. In particular, an element consists of a permutation of $\{1,\dots,n\}$, and then separately for each i, we either choose to send x_i to $x_{\sigma(i)}$ or $1/x_{\sigma(i)}$ (and then $1/x_i$ is sent to the other one). So $|H_n| = 2^n n!$ and is the **hyperoctahedral group** (it's the group of symmetries of an n-dimensional cube). Abstractly, it is the semidirect product $\mathfrak{S}_n \ltimes (\mathbf{Z}/2)^n$ where the generator of the ith copy of $\mathbf{Z}/2$ is the permutation t_i defined by $x_i \mapsto x_i^{-1}$ and $x_j \mapsto x_j$ for $j \ne i$. (In general Lie theory terms, H_n is the Weyl group of $\mathbf{Sp}_{2n}(\mathbf{C})$.)

Lemma 2.4.1. *If ρ is a representation of $\mathbf{Sp}_{2n}(\mathbf{C})$, then $\operatorname{char}(\rho)(x_1,\dots,x_n)$ is invariant under H_n.*

Proof. H_n is generated by $\mathfrak{S}_n$ and $t_1,\dots,t_n$, so it suffices to show invariance under each. For $\sigma \in \mathfrak{S}_n$, we have a $n \times n$ permutation matrix $M(\sigma)$ and conjugating by the following element in $\mathbf{Sp}_{2n}(\mathbf{C})$: $\begin{bmatrix} M(\sigma) & 0 \\ 0 & I'M(\sigma)I' \end{bmatrix}$ has the effect of $x_i^\pm \mapsto x_{\sigma(i)}^\pm$ for a diagonal matrix with entries $x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}$. For t_i, consider the element in $\mathbf{Sp}_{2n}(\mathbf{C})$ given by $e_j \mapsto e_j$ for $j \notin \{i,-i\}$, $e_i \mapsto e_{-i}$, and $e_{-i} \mapsto -e_i$. Then conjugating by this element has the effect of swapping x_i and x_i^{-1} for a diagonal matrix with entries $x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}$. ◻

Corollary 2.4.2. *Every representation of $\mathbf{Sp}_{2n}(\mathbf{C})$ is isomorphic to its dual.*

Proof. If $f(x_1,\dots,x_n)$ is the character of V, then $f(x_1^{-1},\dots,x_n^{-1})$ is the character of V^*, but these are equal by the fact that f is H_n-invariant. ◻

The analogue of $\Lambda(n)$, the ring of symmetric polynomials, is the ring of H_n-invariant functions $\Lambda_{\mathbf{Sp}(2n)} = \mathbf{Z}[x_1^\pm,\dots,x_n^\pm]^{H_n}$. Since characters determine representations, the characters of $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$ are linearly independent. Every monomial is in the orbit of a unique element of the form $x_1^{\lambda_1}\cdots x_n^{\lambda_n}$ for a partition λ, and the characters actually form a basis for $\Lambda_{\mathbf{Sp}(2n)}$. We let $s_{[\lambda]}(x_1,\dots,x_n)$ be the character of $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$.

For reference, we state the Weyl character formula. For $\sigma \in H_n$ we let $\operatorname{sgn}(\sigma)$ denote its sign as an element of $\mathfrak{S}_{2n}$.

Theorem 2.4.3 (Weyl character formula for $\mathbf{Sp}_{2n}(\mathbf{C})$)**.** *Set* $\rho = (n, n-1, \dots, 2, 1)$*. Then*

$$\begin{aligned} s_{[\lambda]}(x_1, \dots, x_n) &= \frac{\det(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)})_{i,j=1}^n}{\prod_{i=1}^n (x_i - x_i^{-1}) \prod_{1 \le i < j \le n} (x_i + x_i^{-1} - x_j - x_j^{-1})} \\ &= \frac{\sum_{\sigma \in H_n} \operatorname{sgn}(\sigma)\sigma(x^{\lambda+\rho})}{\sum_{\sigma \in H_n} \operatorname{sgn}(\sigma)\sigma(x^{\rho})} \end{aligned}$$

Theorem 2.4.4 (Symplectic Cauchy identity)**.**

$$\frac{\prod_{1 \le i < j \le n}(1 - y_i y_j t^2)}{\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} = \sum_\lambda s_{[\lambda]}(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n) t^{|\lambda|}$$

where the sum is over all $\lambda_1 \ge \dots \ge \lambda_n \ge 0$*.*

Proof. Let X be as in the previous section. Then by Corollary 2.2.5, we have

$$\operatorname{char}(\mathbf{C}[X]) = \sum_\lambda s_{[\lambda]}(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n) t^{|\lambda|}$$

Using (3.3.4) and the GL Cauchy identities, we get

$$\operatorname{char}(\mathbf{C}[X]) \prod_{1 \le i < j \le n} (1 - y_i y_j t^2)^{-1} = \prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)^{-1} (1 - x_i^{-1} y_j t)^{-1}$$

(the t^2 is because $\bigwedge^2 \mathbf{C}^n$ are degree 2 functions in $\mathbf{C}[U]$). Rearranging the terms gives us the identity that we want to prove. □

In Theorem 2.3.8, we obtained an identity of the form

$$s_\lambda(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}) = s_{[\lambda]}(x_1, \dots, x_n) + \sum_{\mu,\ |\mu| < |\lambda|} m_{\lambda,\mu} s_{[\mu]}(x_1, \dots, x_n).$$

If we pick an ordering of the partitions of size $\le |\lambda|$ that refines size, i.e., $\alpha < \beta$ implies $|\alpha| \le |\beta|$, then the coefficients $m_{\lambda,\mu}$ can be put into a matrix which is upper-triangular with 1's on the diagonal, and hence is invertible and its inverse again has integer entries. This tells us there are certain integers $n_{\lambda,\mu}$ such that

$$s_{[\lambda]}(x_1, \dots, x_n) = \sum_\mu n_{\lambda,\mu} s_\mu(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}).$$

Furthermore, $n_{\lambda,\lambda} = 1$ and $n_{\lambda,\mu} \ne 0$ implies $|\lambda| \ge |\mu|$. Also, since $m_{\lambda,\mu}$ is independent of n once $n \ge \ell(\lambda)$, the same is true for $n_{\lambda,\mu}$. The coefficients $n_{\lambda,\mu}$ can be determined, but we will postpone it until later (my approach uses Koszul complexes and I will save it for later to minimize outside technicalities).

This leads to the following idea: in the ring of symmetric functions Λ, we define **symplectic Schur functions** via

$$s_{[\lambda]} = \sum_\mu n_{\lambda,\mu} s_\mu.$$

By the upper-triangularity property, these also form a basis for Λ.

We define specialization maps

$$\begin{aligned}\pi_{\mathbf{Sp}(2n)} \colon \Lambda &\to \Lambda_{\mathbf{Sp}(2n)} \\ s_{[\lambda]} &\mapsto \sum_{\mu} n_{\lambda,\mu} s_\mu(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}).\end{aligned}$$

If $n \geq \ell(\lambda)$, then $\pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}) = s_{[\lambda]}(x_1, \dots, x_n)$, but otherwise we do not know. We can be more precise about what happens when $n < \ell(\lambda)$, but again I will postpone the discussion.

Lemma 2.4.5 (Modified symplectic Cauchy identity)**.** *For any n, r, we have*

$$\frac{\prod_{1 \leq i < j \leq r}(1 - y_i y_j t^2)}{\prod_{i=1}^n \prod_{j=1}^r (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} = \sum_\lambda \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}) s_\lambda(y_1, \dots, y_r) t^{|\lambda|}$$

where the sum is over partitions λ with $\ell(\lambda) \leq r$.

Proof. It suffices to show that the identity holds with $\pi_{\mathbf{Sp}(2n)}(s_{[\lambda]})$ replaced by $s_{[\lambda]}(x_1, \dots, x_n)$ and $n \geq 2r$. This is obtained by setting $y_{r+1} = \cdots = y_n = 0$ in the symplectic Cauchy identity. □

We consider the problem of multiplying two symplectic Schur functions. We have integers such that

$$s_{[\mu]} s_{[\nu]} = \sum_\lambda {}^{\mathbf{Sp}}c^\lambda_{\mu,\nu} s_{[\lambda]}$$

We will make use of the following GL-identity:

Corollary 2.4.6 (GL branching rule)**.**

$$s_\lambda(x_1, \dots, x_n, y_1, \dots, y_m) = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu(x_1, \dots, x_n) s_\nu(y_1, \dots, y_m)$$

where the sum is over all pairs of partitions μ, ν.

Proof. Pick r large. We use the Cauchy identity in two ways:

$$\begin{aligned}&\sum_\lambda s_\lambda(x_1, \dots, x_n, y_1, \dots, y_m) s_\lambda(z_1, \dots, z_r) t^{|\lambda|} \\ &= \prod_{i=1}^r \left(\prod_{j=1}^n (1 - x_j z_i t) \prod_{j=1}^m (1 - y_j z_i t) \right) \\ &= \left(\sum_\mu s_\mu(x_1, \dots, x_n) s_\mu(z_1, \dots, z_r) t^{|\mu|} \right) \left(\sum_\nu s_\nu(y_1, \dots, y_m) s_\nu(z_1, \dots, z_r) t^{|\nu|} \right) \\ &= \sum_{\mu,\nu,\lambda} s_\mu(x_1, \dots, x_n) s_\nu(y_1, \dots, y_m) c^\lambda_{\mu,\nu} s_\lambda(z_1, \dots, z_r) t^{|\lambda|}\end{aligned}$$

By linear independence of Schur polynomials, the coefficients of $s_\lambda(z_1, \dots, z_r)$ are the same, which is the identity we want to prove. □

Theorem 2.4.7 (Newell–Littlewood product formula)**.** *We have*

$$s_{[\mu]}(x_1, \dots, x_n) s_{[\nu]}(x_1, \dots, x_n) = \sum_{\lambda,\alpha,\beta,\gamma} c^\lambda_{\beta,\gamma} c^\mu_{\beta,\alpha} c^\nu_{\gamma,\alpha} \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]})$$

and, in particular,

$$ {}^{\mathbf{Sp}}c^{\lambda}_{\mu,\nu} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha,\beta} c^{\nu}_{\alpha,\gamma} c^{\lambda}_{\beta,\gamma} $$

where the sum is over all triples of partitions.

Proof. Using the modified symplectic Cauchy identity and the usual Cauchy identity, we have

$$ \frac{\prod_{1 \le i < j \le 2n}(1 - y_i y_j t^2)}{\prod_{i=1}^n \prod_{j=1}^{2n} (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} \cdot \prod_{i=1}^{n} \prod_{j=n+1}^{2n} (1 - y_i y_j t^2)^{-1} $$

$$ = \left(\sum_{\lambda} \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}) s_\lambda(y_1, \dots, y_{2n}) t^{|\lambda|} \right) \left(\sum_{\alpha} s_\alpha(y_1, \dots, y_n) s_\alpha(y_{n+1}, \dots, y_{2n}) t^{2|\alpha|} \right) $$

$$ = \left(\sum_{\lambda,\mu,\nu} \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}) c^{\lambda}_{\beta,\gamma} s_\beta(y_1, \dots, y_n) s_\gamma(y_{n+1}, \dots, y_{2n}) t^{|\beta|+|\gamma|} \right) \left(\sum_{\alpha} s_\alpha(y_1, \dots, y_n) s_\alpha(y_{n+1}, \dots, y_{2n}) t^{2|\alpha|} \right) $$

$$ = \sum_{\lambda,\beta,\gamma,\alpha,\mu,\nu} \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}) c^{\lambda}_{\beta,\gamma} c^{\mu}_{\beta,\alpha} c^{\nu}_{\gamma,\alpha} s_\mu(y_1, \dots, y_n) s_\nu(y_{n+1}, \dots, y_{2n}) t^{|\mu|+|\nu|}. $$

But also, the first expression factors as

$$ \frac{\prod_{1 \le i < j \le n}(1 - y_i y_j t)}{\prod_{i=1}^n \prod_{j=1}^{n} (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} \cdot \frac{\prod_{n+1 \le i < j \le 2n}(1 - y_i y_j t)}{\prod_{i=1}^n \prod_{j=n+1}^{2n} (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} $$

$$ = \left(\sum_{\mu} s_{[\mu]}(x_1, \dots, x_n) s_\mu(y_1, \dots, y_n) t^{|\mu|} \right) \left(\sum_{\nu} s_{[\nu]}(x_1, \dots, x_n) s_\nu(y_{n+1}, \dots, y_{2n}) t^{|\nu|} \right) $$

and compare the coefficients of $s_\mu(y_1, \dots, y_n) s_\nu(y_{n+1}, \dots, y_{2n})$ to get

$$ s_{[\mu]}(x_1, \dots, x_n) s_{[\nu]}(x_1, \dots, x_n) = \sum_{\lambda,\alpha,\beta,\gamma} c^{\lambda}_{\beta,\gamma} c^{\mu}_{\beta,\alpha} c^{\nu}_{\gamma,\alpha} \pi_{\mathbf{Sp}(2n)}(s_{[\lambda]}). \qquad \square $$

3. Orthogonal groups

We now deal with the next series, the orthogonal groups. Many of the proofs are the same as for the symplectic groups, so we will just go through the definitions and highlight the differences.

3.1. **Definitions and basic properties.** We defined an action of $\mathbf{GL}_m(\mathbf{C})$ on $\mathrm{Sym}^2 \mathbf{C}^m$, the space of $m \times m$ symmetric matrices, by

$$ g \cdot u = (g^{-1})^T u g^{-1}. $$

The **orthogonal group** $\mathbf{O}_m(\mathbf{C})$ is the stabilizer of a full rank matrix. Note that any two full rank matrices are in the same orbit, so any two choices lead to conjugate subgroups of $\mathbf{GL}_m(\mathbf{C})$, and in particular are isomorphic.

For concreteness, let I' be the $m \times m$ matrix which is 1 on the antidiagonal, i.e., $I'_{i,j} = 1$ if $j = m + 1 - i$ and 0 otherwise. We take this as our full symmetric matrix. In symbols,

$$ \mathbf{O}_m(\mathbf{C}) = \{g \mid g^T I' g = I'\} = \{g \in \mathbf{GL}_m(\mathbf{C}) \mid I'(g^T)^{-1} I' = g\}. $$

If $g \in \mathbf{O}_m(\mathbf{C})$, then $\det g \in \{1, -1\}$ (and both values are possible). We define the **special orthogonal group** as those with determinant 1:

$$\mathbf{SO}_m(\mathbf{C}) = \{g \in \mathbf{O}_m(\mathbf{C}) \mid \det g = 1\}.$$

Unlike the symplectic case, we do not need to restrict the parity of m (although there will be differences in the even and odd cases). Note that either description shows that $\mathbf{O}_m(\mathbf{C})$ is the zero set of polynomial functions on the space of $m \times m$ matrices, and hence is an affine variety (since we can realize $\mathbf{GL}$ as an affine variety). We will write $m = 2n$ in the even case and $m = 2n+1$ in the odd case. If we want to treat both cases uniformly, we will write $m = 2n + \varepsilon$ where $\varepsilon \in \{0, 1\}$.

We can compute the Krull dimension:

$$\dim \mathbf{O}_m(\mathbf{C}) = \dim \mathbf{GL}_m(\mathbf{C}) - \dim \operatorname{Sym}^2 \mathbf{C}^m = m^2 - \binom{m+1}{2} = \binom{m}{2}.$$

The matrix I' defines a bilinear form on $\mathbf{C}^m$ as follows: given $v, w \in \mathbf{C}^m$, we define

$$\omega(v, w) = v^T I' w.$$

Since $I'^T = I'$, we see that $\omega(v, w) = \omega(w, v)$, so it is symmetric. Furthermore, since $\operatorname{rank} I' = m$, ω is **non-degenerate**: for all v, there exists w such that $\omega(v, w) \neq 0$. An **orthogonal form** is, by definition, a symmetric non-degenerate bilinear form. With respect to the standard basis $e_1, \dots, e_m$, we have

$$\omega(e_i, e_{m+1-j}) = \delta_{i,j}.$$

We will call any basis with this property a **hyperbolic basis**.

A representation is a homomorphism ρ from either $\mathbf{SO}_m(\mathbf{C})$ or $\mathbf{O}_m(\mathbf{C})$ to $\mathbf{GL}(W)$ for some vector space W whose entries are rational functions after picking a basis for W. Much of the theory from $\mathbf{GL}_m(\mathbf{C})$ carries over. We repeat it here with the updated terms. The representations of both groups are related, though in a slightly tedious way. We will focus on $\mathbf{SO}_m(\mathbf{C})$.

Given a hyperbolic basis, we let the Borel subgroup B be the group of upper triangular matrices in $\mathbf{SO}_m(\mathbf{C})$ and the maximal torus T be the group of diagonal matrices. The latter are diagonal matrices with entries $x_1, x_2, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$ if $m = 2n$ is even and $x_1, x_2, \dots, x_n, 1, x_n^{-1}, \dots, x_1^{-1}$ if $m = 2n+1$ is odd.

If $\rho \colon \mathbf{SO}_m(\mathbf{C}) \to \mathbf{GL}(W)$ is a representation, then there is a basis $w_1, \dots, w_r$ for W such that each w_i is an eigenvector of $\rho(t)$ for all $t \in T$; if the entries of t are $x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}$, then the eigenvalues are of the form $x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$ for some $\mu_{i,1}, \dots, \mu_{i,n} \in \mathbf{Z}$. Any vector w which is an eigenvector for all $\rho(t)$ is called a **weight vector** and $(\mu_1, \dots, \mu_n)$ is its **weight** if $\rho(t)(w) = x_1^{\mu_1} \cdots x_n^{\mu_n} w$ for all $x_1, \dots, x_n$. We also write $\mu(t)$ for $x_1^{\mu_1} \cdots x_n^{\mu_n}$.

The **character** of ρ is defined to be the function $\operatorname{char}(\rho)(x_1, \dots, x_n) = \operatorname{Tr} \rho(t)$ where t is the diagonal matrix with entries $x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}$ and Tr denotes trace. Alternatively, $\operatorname{char}(\rho)(x_1, \dots, x_n) = \sum_{i=1}^r x_1^{\mu_{i,1}} \cdots x_n^{\mu_{i,n}}$.

A nonzero vector $w \in W$ is a **highest weight vector** if $b \cdot w$ is a scalar multiple of w for all $b \in B$. In particular, it is an eigenvector for all $\rho(t)$ for $t \in T$, so it is also a weight vector. If μ is the weight of this vector, we write $\mu(b)$ for the scalar multiple i.e., $b \cdot w = \mu(b) w$.

We can define direct sums, tensor products, and duals in the same way and they transform characters in the same way.

Theorem 3.1.1. *(1) Every finite-dimensional representation of $\mathbf{SO}_m(\mathbf{C})$ is semisimple.*

(2) *Two representations of* $\mathbf{SO}_m(\mathbf{C})$ *are isomorphic if and only if they have the same character.*
(3) *(Schur's lemma) There are no nonzero homomorphisms between non-isomorphic simple representations. Any homomorphism from a simple representation to itself must be a scalar multiple of the identity.*
(4) *Every finite-dimensional representation contains a highest weight vector.*
(5) *Any two highest weight vectors (for a particular choice of Borel subgroup) in an irreducible representation are scalar multiples of each other.*
(6) *If* m *is odd, then the weight* $(\mu_1, \dots, \mu_n)$ *of a highest weight vector satisfies* $\mu_1 \geq \cdots \geq \mu_n \geq 0$. *If* m *is even, then* $\mu_1 \geq \cdots \geq \mu_{n-1} \geq |\mu_n|$.
(7) *For every* $(\mu_1, \dots, \mu_n) \in \mathbf{Z}^n$ *satisfying the conditions above, there is an irreducible representation whose highest weight vector has that weight.*

When $\mu_n = 0$ or m is odd, we will denote the irreducible representation whose highest weight is μ by $\mathbf{S}_{[\mu]}(\mathbf{C}^m)$ (not to be confused with irreducible representations of the symplectic group). Otherwise, when m is even and $\mu_n > 0$, we use $\mathbf{S}_{[\mu]^+}(\mathbf{C}^{2n})$ to denote the representation with highest weight μ and $\mathbf{S}_{[\mu]^-}(\mathbf{C}^{2n})$ to denote the representation with highest weight $(\mu_1, \dots, \mu_{n-1}, -\mu_n)$. We reserve $\mathbf{S}_{[\mu]}(\mathbf{C}^{2n})$ for their direct sum (this will make statements easier).

Example 3.1.2. (1) Take ρ to be the natural inclusion $\mathbf{SO}_m(\mathbf{C}) \to \mathbf{GL}_m(\mathbf{C})$, so $W = \mathbf{C}^m$. The standard basis vectors $e_1, \dots, e_m$ are weight vectors where for $i = 1, \dots, n$, the weight of e_i is $(0, \dots, 1, \dots, 0)$ (with a 1 in position i) and the weight of e_{m+1-i} is $(0, \dots, -1, \dots, 0)$ (with a -1 in position i). If m is odd, the weight of e_{n+1} is $(0, 0, \dots, 0)$. The highest weight vector is e_1 and the character is $\sum_{i=1}^n (x_i + x_i^{-1}) + \varepsilon$. Note that this implies that W is isomorphic to its dual representation since its character is invariant under $x_i \mapsto x_i^{-1}$.

(2) In general, if $\rho \colon \mathbf{GL}_m(\mathbf{C}) \to \mathbf{GL}(W)$ is a rational representation, then we get a rational representation by restricting to $\mathbf{SO}_m(\mathbf{C})$. We denote this representation by $\rho|_{\mathbf{SO}_m(\mathbf{C})}$. Then we have

$$\mathrm{char}(\rho|_{\mathbf{SO}_m(\mathbf{C})})(x_1, \dots, x_n) = \mathrm{char}(\rho)(x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}).$$

(3) Consider the second symmetric power $\mathrm{Sym}^2 \mathbf{C}^m$ with $m \geq 2$. Its character as a $\mathbf{GL}_m(\mathbf{C})$ representation (using y variables) is $\sum_{1 \leq i \leq j \leq m} y_i y_j$. So the character of its restriction to $\mathbf{SO}_m(\mathbf{C})$ is $\sum_{1 \leq i < j \leq n} (x_i x_j + x_i^{-1} x_j^{-1}) + \sum_{i \neq j} x_i x_j^{-1} + \varepsilon \sum_{i=1}^n x_i + m$. By definition of the orthogonal group, the line spanned by I' is a trivial subrepresentation of $\mathrm{Sym}^2 \mathbf{C}^m$. Let $\mathrm{Sym}_0^2 \mathbf{C}^m$ denote a complementary subrepresentation. In fact, this is irreducible: the only weights that appear in it that potentially be highest weights are $(2, 0, \dots, 0)$ and $(0, 0, \dots, 0)$. We leave it as an exercise to show that there are no further trivial subrepresentations. □

As before, we will address the question: given an irreducible $\mathbf{GL}_m(\mathbf{C})$ representation $\mathbf{S}_\lambda(\mathbf{C}^m)$, how do we determine the decomposition of its restriction to $\mathbf{SO}_m(\mathbf{C})$ as a direct sum of irreducible representations? We will see that the answer is uniform in m as long as $n \geq \ell(\lambda)$ (with some minor complications when m is even), but otherwise there are complications (which we will also discuss).

The following fact will be used and we sketch a proof in the homework:

Proposition 3.1.3. $\mathbf{SO}_m(\mathbf{C})$ *is connected in the Zariski topology.*

Note, however, that $\mathbf{O}_m(\mathbf{C})$ has 2 connected components, corresponding to whether the determinant of an element is 1 or -1.

Remark 3.1.4. $\mathbf{SO}_m(\mathbf{C})$ is a normal subgroup of $\mathbf{O}_m(\mathbf{C})$ and the quotient is isomorphic to $\mathbf{Z}/2$.

When m is odd, $-I \in \mathbf{O}_m(\mathbf{C})$ has determinant -1 and hence is a representative for the nontrivial coset. Since it is in the center of $\mathbf{O}_m(\mathbf{C})$, we have $\mathbf{O}_m(\mathbf{C}) \cong \mathbf{SO}_m(\mathbf{C}) \times \mathbf{Z}/2$. This makes it easy to understand the irreducible representations of $\mathbf{O}_m(\mathbf{C})$ in terms of those of $\mathbf{SO}_m(\mathbf{C})$: we take all of them and either have $-I$ acts as 1 or -1 (so we get twice as many representations).

The situation when m is even is more complicated: $\mathbf{O}_{2n}(\mathbf{C})$ is not a direct product and the relationship between irreducible representations of $\mathbf{SO}_{2n}(\mathbf{C})$ and $\mathbf{O}_{2n}(\mathbf{C})$ is more subtle. For every partition λ with $\ell(\lambda) \le n-1$, there are a pair of irreducible representations V_λ and $V_\lambda \otimes \det$ which differ by tensoring with the determinant representation both of which are isomorphic to $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$ when restricted to $\mathbf{SO}_{2n}(\mathbf{C})$. For each partition λ with $\ell(\lambda) = n$, there is one irreducible representation $V_\lambda \cong V_\lambda \otimes \det$ whose restriction to $\mathbf{SO}_{2n}(\mathbf{C})$ is isomorphic to $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n}) = \mathbf{S}_{[\lambda]^+}(\mathbf{C}^{2n}) \oplus \mathbf{S}_{[\lambda]^-}(\mathbf{C}^{2n})$. □

Remark 3.1.5. For those who have studied Lie algebras: every representation of a Lie group can be differentiated to get a representation of the corresponding Lie algebra. The reverse is almost true. Precisely, every finite-dimensional representation of the Lie algebra (in our examples) can be integrated to a representation of the *simply-connected cover* of the Lie group. Here simply-connected is understood with respect to the standard Euclidean topology, i.e., the fundamental group is trivial, or all loops are homotopy equivalent to a point. Both $\mathbf{GL}_n(\mathbf{C})$ and $\mathbf{Sp}_{2n}(\mathbf{C})$ are simply-connected, but $\mathbf{SO}_m(\mathbf{C})$ is not: its fundamental group is isomorphic to $\mathbf{Z}/2$. So it has a double cover, which is called the spin group. This means that $\mathbf{SO}_m(\mathbf{C})$ only sees roughly "half" of the representations of its Lie algebra. Time permitting, we will discuss the spin group and the missing representations later in the course. □

3.2. **Multiplicity-free action.** We will study a multiplicity-free action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{SO}_m(\mathbf{C})$. This representation will be very helpful for understanding the branching problem. The results about representations carry over to this product group so we won't restate them.

Given an orthogonal form ω on $\mathbf{C}^m$, a subspace $V \subset \mathbf{C}^m$ is **isotropic** if $\omega(x, y) = 0$ for all $x, y \in V$. A few basic properties (which we leave to exercises):

(1) If V is isotropic, then $\dim V \le n$.
(2) Given 2 isotropic subspaces V_1, V_2 with $\dim V_1 = \dim V_2$, there exists $g \in \mathbf{O}_m(\mathbf{C})$ such that $gV_1 = V_2$. Assuming that either m is odd, or that m is even and $\dim V_i < n$, we can actually find $g \in \mathbf{SO}_m(\mathbf{C})$ such that $gV_1 = V_2$. In the exceptional case that m is even and $\dim V_i = n$, there are 2 orbits of isotropic subspaces under the action of $\mathbf{SO}_{2n}(\mathbf{C})$. In particular, the span of $e_1, \dots, e_n$ and $e_1, \dots, e_{n-1}, e_{n+1}$ are in separate orbits.
(3) Every isotropic subspace is contained in an n-dimensional isotropic subspace.

Now consider the space $U = (\mathbf{C}^m \otimes \mathbf{C}^n)^*$ which we identify with $m \times n$ matrices. We define an action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{O}_m(\mathbf{C})$ by

$$(g, h) \cdot u = (h^{-1})^T u g^{-1}.$$

Define

$$X = \{u \in U \mid u^T I' u = 0\}.$$

Since $u^T I' u$ is a symmetric matrix whose entries are polynomial functions of the entries of u, we see that X is an affine variety which is the solution set of $\binom{n+1}{2}$ polynomials. Alternatively, X is the subset of maps $u\colon \mathbf{C}^n \to \mathbf{C}^m$ such that $u(\mathbf{C}^n)$ is an isotropic subspace. Then X is closed under the action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{O}_m(\mathbf{C})$: if $u \in X$, then using that $(h^{-1})^T = I'hI'$, we have

$$(g^{-1})^T u^T I' h^T I' I' I' h I' u g^{-1} = (g^{-1})^T u^T I' u g^{-1} = (g^{-1})^T 0 g^{-1} = 0.$$

Proposition 3.2.1. *X is irreducible if m is odd and has 2 irreducible components if m is even.*

Proof. Let X° be the subset of matrices of full rank n, i.e., where the map $u\colon \mathbf{C}^n \to \mathbf{C}^m$ is injective. Then $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{O}_m(\mathbf{C})$ acts transitively on X°: the image is an n-dimensional subspace and $\mathbf{O}_m(\mathbf{C})$ acts transitively on those, while for a fixed n-dimensional isotropic subspace L, the set of linear isomorphisms $\mathbf{C}^n \to L$ is a single orbit under pre-composition by $\mathbf{GL}_n(\mathbf{C})$. If m is odd, then we can replace $\mathbf{O}_m(\mathbf{C})$ by $\mathbf{SO}_m(\mathbf{C})$.

Picking any point in $x \in X^\circ$, we get a surjective map $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{O}_m(\mathbf{C}) \to X^\circ$ via $(g,h) \mapsto (g,h)\cdot x$ which remains surjective when m is odd when we use $\mathbf{SO}_m(\mathbf{C})$. Since $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{SO}_m(\mathbf{C})$ is connected, it is also irreducible. This implies that the target is also irreducible when m is odd. Otherwise, when n is even, we see that the source has 2 connected components, and hence 2 irreducible components, and hence the target has at most 2 irreducible components. These actually have to be disjoint: if not, then by considering a point in the intersection, we would see that all maximal isotropic subspaces are in the same $\mathbf{SO}_{2n}(\mathbf{C})$ orbit which is false.

Next, we show that X° is dense in X which finishes the proof. This is the same as the proof of Proposition 2.2.1. ☐

Let J be the $m \times n$ matrix with $J_{i,i} = 1$ for $i = 1, \dots, n$ and 0's elsewhere. If m is even, we also define J' to be the $2n \times n$ matrix with $J_{i,i} = 1$ for $i = 1, \dots, n-1$ and $J_{n+1,n} = 1$ and 0's elsewhere. Let $B \subset \mathbf{GL}_n(\mathbf{C})$ be the subgroup of upper-triangular matrices, and similarly, let $B' \subset \mathbf{SO}_m(\mathbf{C})$ be the subgroup of upper-triangular matrices.

Let $f_i \in \mathbf{C}[X]$ be the function which takes the determinant of the upper-left $i \times i$ submatrix. If m is even, define $f_n' \in \mathbf{C}[X]$ to be the determinant of the $n \times n$ submatrix with rows $1, 2, \dots, n-1, n+1$.

Proposition 3.2.2. *f_i is a highest weight vector with weight $(1, \dots, 1, 0, \dots, 0), (1, \dots, 1, 0, \dots, 0)$ (the number of 1's in each vector is i) and f_n' is a highest weight vector with weight $(1, 1, \dots, 1)$, $(1, \dots, 1, -1)$.*

Lemma 3.2.3. *The $B \times B'$ orbit in X containing J is open. If m is odd, it is also dense, otherwise it is dense in the irreducible component containing it. The other component contains the $B \times B'$ orbit containing J' as a dense orbit.*

Proof. The orbit of J is the set of matrices with $f_1, \dots, f_n \neq 0$ and the orbit of J' is the same with f_n replaced by f_n'. The ideas are similar to the proof in the symplectic case. ☐

Lemma 3.2.4. *If (λ, λ') is the weight of a highest weight vector in $\mathbf{C}[X]$, then $\lambda_i = \lambda_i'$ for $1 \le i \le n-1$. If m is odd, then furthermore $\lambda_n = \lambda_n'$ and if m is even, then $\lambda_n = |\lambda_n'|$.*

Proof. If m is odd, then the stabilizer of J contains pairs of diagonal matrices (g,h) where the entries of h are $x_1, \dots, x_n, 1, x_n^{-1}, \dots, x_1^{-1}$ and the entries of g are $x_1^{-1}, \dots, x_n^{-1}$. By

Theorem 1.2.6, we see that $x_1^{\lambda_1-\lambda'_1} \cdots x_n^{\lambda_n-\lambda'_n} = 1$ for all $x_1, \dots, x_m$. This forces all of the exponents to be 0.

If m is even, the stabilizer of J contains pairs of diagonal matrices (g, h) where the entries of h are $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$ and the entries of g are $x_1^{-1}, \dots, x_n^{-1}$. So any highest weight appearing in the functions on the irreducible component containing J satisfies $\lambda_i = \lambda'_i$ for $i = 1, \dots, n$. Similarly, the stabilizer of J contains pairs of diagonal matrices (g, h) where the entries of h are $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$ and the entries of g are $x_1^{-1}, \dots, x_{n-1}^{-1}, x_n$. So any highest weight appearing in the functions on the irreducible component containing J' satisfies $\lambda_i = \lambda'_i$ for $i = 1, \dots, n-1$ and $\lambda_n = -\lambda'_n$. □

Finally, we use Proposition 1.2.7 to realize the highest weights above. For m odd this is straightforward as X is irreducible and every partition is a sum of vectors of the form $(1, \dots, 1, 0, \dots, 0)$. When m is even, we work with the functions on each irreducible component. The one containing J has functions with highest weights (λ, λ') with $\lambda = \lambda'$ and the one containing J' has functions with highest weights (λ, λ') with $\lambda_i = \lambda'_i$ for $i = 1, \dots, n-1$ and $\lambda_n = -\lambda'_n$ for the same reason. Hence X has functions with those highest weights as well. The only thing we have to be careful of is that when $\lambda_n = \lambda'_n = 0$, these highest weights may appear with multiplicity 2 since we get one contribution from each component. But this is easy to rule out using their explicit form as products of $f_1, \dots, f_{n-1}$.

Corollary 3.2.5 (Orthogonal Cauchy identity)**.** *We have an isomorphism of* $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{SO}_m(\mathbf{C})$ *representations*

$$\mathbf{C}[X] \cong \bigoplus_{\lambda} \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{[\lambda]}(\mathbf{C}^m)$$

where the sum is over all integer partitions $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$.

3.3. **Branching problem.** We can use X from the previous section to study the branching problem from $\mathbf{GL}_m(\mathbf{C})$ to $\mathbf{SO}_m(\mathbf{C})$.

First, we revisit the equations defining X. We said $X = \{u \mid u^T I' u = 0\}$. Since $I' = I'$, $u^T I' u$ is a symmetric $n \times n$ matrix. So taking its upper-triangular entries, we get $\binom{n+1}{2}$ equations whose common solution set is X.

Proposition 3.3.1. $\dim X = \dim U - \binom{n+1}{2}$.

Let's unpackage Corollary 2.3.3 for our example. We take U as in this section and $f_1, \dots, f_r$ are the entries of the skew-symmetric matrix $u^T I' u$ with $r = \binom{n+1}{2}$. From the calculation in the previous section, we get

$$((g, h) \cdot u)^T I'((g, h) \cdot u) = (g^{-1})^T (u^T I' u) g^{-1}$$

so that we see that $f_1, \dots, f_r$ span a subrepresentation V of $\mathbf{C}[U]$ where $V \cong \operatorname{Sym}^2 \mathbf{C}^n$ (with the $\mathbf{SO}_m(\mathbf{C})$ factor acting trivially). This implies that we have an isomorphism of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{SO}_m(\mathbf{C})$ representations

$$\mathbf{C}[U]/(f_1, \dots, f_r) \otimes \operatorname{Sym}(\operatorname{Sym}^2 \mathbf{C}^n) \cong \mathbf{C}[U].$$

We want to identify $\mathbf{C}[U]/(f_1, \dots, f_r)$ with $\mathbf{C}[X]$. As in the symplectic case, we have (using Theorem 2.3.5):

Proposition 3.3.2. *Any polynomial that is identically* 0 *on* X *is generated by the entries of* $u^T I' u$*. In particular,* $\mathbf{C}[X] = \mathbf{C}[U]/(f_1, \dots, f_r)$.

Remark 3.3.3 (Weyl's construction)**.** $\mathbf{C}[X]$ is a quotient of $\mathbf{C}[U]$ and the kernel is generated by $\mathrm{Sym}^2(\mathbf{C}^n)$, so in particular the kernel is a quotient $\mathbf{C}[U]\otimes\mathrm{Sym}^2\mathbf{C}^n$, i.e., we have an exact sequence

$$\mathbf{C}[U]\otimes\mathrm{Sym}^2\mathbf{C}^n\to\mathbf{C}[U]\to\mathbf{C}[X]\to 0.$$

We can consider the $\mathbf{S}_\lambda(\mathbf{C}^n)$ components of this sequence to get

$$\bigoplus_\mu \mathbf{S}_\mu(\mathbf{C}^m)^{\oplus c^\lambda_{\mu,(2)}}\to\mathbf{S}_\lambda(\mathbf{C}^m)\to\mathbf{S}_{[\lambda]}(\mathbf{C}^m)\to 0.$$

The first term simplifies a bit: we just sum over μ such that $|\lambda|=|\mu|+2$ and λ/μ is a horizontal strip. This gives us a recipe for constructing $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$ using Schur functors: given μ and λ as above, we have a map

$$\mathbf{S}_\mu(\mathbf{C}^m)\otimes\mathrm{Sym}^2(\mathbf{C}^m)\to\mathbf{S}_\lambda(\mathbf{C}^m)$$

by Pieri's rule. We can identify $\mathbf{S}_\mu(\mathbf{C}^m)$ as the subaspace $\mathbf{S}_\mu(\mathbf{C}^m)\otimes\langle I'\rangle$ of the source, and then the image of this subspace is a $\mathbf{O}_m(\mathbf{C})$-subrepresentation of $\mathbf{S}_\lambda(\mathbf{C}^m)$. If we quotient by all of these subrepresentations, then we get $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$. This is also known as the space of "traceless tensors". □

In our situation, we get

$$\mathbf{C}[X]\otimes\mathrm{Sym}(\mathrm{Sym}^2\mathbf{C}^n)\cong\mathrm{Sym}(\mathbf{C}^n\otimes\mathbf{C}^m). \tag{3.3.4}$$

Using all of the various multiplicity-free representations we've studied this becomes the following identity:

$$\left(\bigoplus_\mu \mathbf{S}_\mu(\mathbf{C}^n)\otimes\mathbf{S}_{[\mu]}(\mathbf{C}^m)\right)\otimes\left(\bigoplus_\nu \mathbf{S}_{2\nu}(\mathbf{C}^n)\right)\cong\bigoplus_\lambda \mathbf{S}_\lambda(\mathbf{C}^n)\otimes\mathbf{S}_\lambda(\mathbf{C}^m).$$

(in the last $\mathbf{S}_\lambda(\mathbf{C}^{2n})$, this is the restriction of the Schur functor to $\mathbf{SO}_m(\mathbf{C})$.) The left hand side can be combined using Littlewood–Richardson coefficients:

$$\bigoplus_{\mu,\nu,\lambda}(\mathbf{S}_\lambda(\mathbf{C}^n)\otimes\mathbf{S}_{[\mu]}(\mathbf{C}^m))^{\oplus c^\lambda_{\mu,2\nu}}.$$

Comparing these two expressions we see how Schur functors decompose into irreducible representations:

Theorem 3.3.5 (Stable branching rule from $\mathbf{GL}$ to $\mathbf{O}$)**.** *Let λ be a partition with $\ell(\lambda)\le n$. Then we have an isomorphism of $\mathbf{O}_m(\mathbf{C})$-representations*

$$\mathbf{S}_\lambda(\mathbf{C}^m)\cong\bigoplus_\mu \mathbf{S}_{[\mu]}(\mathbf{C}^m)^{\oplus m_{\lambda,\mu}},\qquad m_{\lambda,\mu}=\sum_\nu c^\lambda_{\mu,2\nu}.$$

The "stable" in the name refers to the condition $\ell(\lambda)\le n$. Note that the answer does not depend on n as long as this condition is satisfied. We can make sense of this question in general, but the method above does not give an answer. We call the case $\ell(\lambda)>n$ the unstable case.

Example 3.3.6. When $\lambda=\mu$, we have $m_{\lambda,\lambda}=1$ since the only valid choice for ν is the empty partition corresponding to the trivial representation, and $c^\lambda_{\lambda,\varnothing}=1$. The highest weight vector for $\mathbf{GL}_m(\mathbf{C})$ in $\mathbf{S}_\lambda(\mathbf{C}^m)$ is also a highest weight vector of weight λ for the subgroup $\mathbf{SO}_m(\mathbf{C})$. □

Example 3.3.7. If $\lambda = (d)$, then $c^\lambda_{\mu,2\nu} > 0$ implies that both μ and 2ν are contained in (d), i.e., $Y(\mu)$ is a row of size $\le d$ and 2ν is an even length row of size $\le d$. So if $1 \le n$, we see that

$$\mathrm{Sym}^d(\mathbf{C}^m) \cong \mathbf{S}_{[d]}(\mathbf{C}^m) \oplus \mathbf{S}_{[d-2]}(\mathbf{C}^m) \oplus \mathbf{S}_{[d-4]}(\mathbf{C}^m) \oplus \cdots . \qquad \square$$

3.4. **Characters of the orthogonal group.** Given a representation $\rho\colon \mathbf{SO}_m(\mathbf{C}) \to \mathbf{GL}(W)$, its character $\mathrm{char}(\rho)(x_1, \dots, x_n)$ is the trace of the diagonal matrix with entries $x_1, \dots, x_n$, (1?), $x_n^{-1}, \dots, x_1^{-1}$. Recall that H_n is the hyperoctahedral group acting on the variables by permutations and inversions. We let H'_n be the subgroup of H_n consisting of elements that do an even number of inversions, i.e., the number of i such that $\sigma(x_i) = x_j^{-1}$ is even.

Lemma 3.4.1. *Let ρ be a representation of $\mathbf{SO}_m(\mathbf{C})$. If m is odd, then $\mathrm{char}(\rho)(x_1, \dots, x_n)$ is invariant under H_n. If m is even, then $\mathrm{char}(\rho)(x_1, \dots, x_n)$ is invariant under H'_n.*

Proof. H_n is generated by $\mathfrak{S}_n$ and $t_1, \dots, t_n$, so it suffices to show invariance under each in the odd case and invariance under pairwise products of the t_i in the even case. For $\sigma \in \mathfrak{S}_n$, we have a $n \times n$ permutation matrix $M(\sigma)$ and conjugating by the following element in $\mathbf{SO}_{2n}(\mathbf{C})$: $\begin{bmatrix} M(\sigma) & 0 \\ 0 & I'M(\sigma)I' \end{bmatrix}$ has the effect of $x_i^\pm \mapsto x_{\sigma(i)}^\pm$ for a diagonal matrix with entries $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$. For $m = 2n+1$ we take instead the matrix $\begin{bmatrix} M(\sigma) & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I'M(\sigma)I' \end{bmatrix}$.

For t_i, consider the element in $\mathbf{O}_m(\mathbf{C})$ given by $e_j \mapsto e_j$ for $j \notin \{i, -i\}$, $e_i \mapsto e_{-i}$, and $e_{-i} \mapsto e_i$. Then conjugating by this element has the effect of swapping x_i and x_i^{-1} for a diagonal matrix with entries $x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}$. This has determinant -1, so in the odd case, we can get an element of $\mathbf{SO}_{2n+1}(\mathbf{C})$ which is the same except $e_{n+1} \mapsto -e_{n+1}$. In the even case, this shows that products of the pairwise t_i are elements of $\mathbf{SO}_{2n}(\mathbf{C})$ since they have determinant 1. $\square$

Corollary 3.4.2. *If m is odd or divisible by 4, then every representation of $\mathbf{SO}_m(\mathbf{C})$ is isomorphic to its dual.*

Proof. If $f(x_1, \dots, x_n)$ is the character of V, then $f(x_1^{-1}, \dots, x_n^{-1})$ is the character of V^*, but these are equal by the fact that f is H_n-invariant (in the odd case) or H'_n (in the case where n is even). $\square$

Remark 3.4.3. How about if $m = 2 \pmod 4$? Actually, the same argument works to show that $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$ is self-dual for all partitions λ since these are all $\mathbf{O}_m(\mathbf{C})$-representations, and we can realize conjugation by all elements of H_n there. But recall that $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$ is the sum of two irreducible representations as a $\mathbf{SO}_m(\mathbf{C})$-representation. When n is odd, these are not self-dual, but are duals of each other (we will not prove this). $\square$

The analogue of $\Lambda(n)$, the ring of symmetric polynomials, is $\Lambda_{\mathbf{SO}(2n+1)} = \mathbf{Z}[x_1^\pm, \dots, x_n^\pm]^{H_n}$ and $\Lambda_{\mathbf{SO}(2n+1)} = \mathbf{Z}[x_1^\pm, \dots, x_n^\pm]^{H'_n}$. However, it will be better to work with something slightly more uniform, so we define $\Lambda_{\mathbf{O}(m)} = \mathbf{Z}[x_1^\pm, \dots, x_n^\pm]^{H_n}$. This is not literally the space where characters of $\mathbf{O}(m)$ live since we are ignoring the determinant representation. Since characters determine representations, the characters of $\mathbf{S}_{[\lambda]}(\mathbf{C}^{2n})$ are linearly independent. Every monomial is in the orbit of a unique element of the form $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$ for a partition λ, and the characters actually form a basis for $\Lambda_{\mathbf{Sp}}(n)$. We let $s_{[\lambda]}(x_1, \dots, x_n)$ be the character of $\mathbf{S}_{[\lambda]}(\mathbf{C}^m)$.

For reference, we state the Weyl character formula. As before, for $\sigma \in H_n$ we let $\mathrm{sgn}(\sigma)$ denote its sign as an element of $\mathfrak{S}_{2n}$.

Theorem 3.4.4 (Weyl character formula for $\mathbf{SO}_{2n+1}(\mathbf{C})$)**.** *Set* $\rho = (n - \frac{1}{2}, n - \frac{3}{2}, \dots, \frac{3}{2}, \frac{1}{2})$. *Then*

$$\begin{aligned} s_{[\lambda]}(x_1, \dots, x_n) &= \frac{\det(x_i^{\lambda_j + n - j + \frac{1}{2}} - x_i^{-(\lambda_j + n - j + \frac{1}{2})})_{i,j=1}^n}{\prod_{i=1}^n (x_i^{1/2} - x_i^{-1/2}) \prod_{1 \le i < j \le n} (x_i + x_i^{-1} - x_j - x_j^{-1})} \\ &= \frac{\sum_{\sigma \in H_n} \mathrm{sgn}(\sigma) \sigma(x^{\lambda+\rho})}{\sum_{\sigma \in H_n} \mathrm{sgn}(\sigma)\sigma(x^\rho)} \end{aligned}$$

The case of $\mathbf{SO}_{2n}(\mathbf{C})$ is a bit more complicated, so we will omit it.

Theorem 3.4.5 (Orthogonal Cauchy identity)**.** *For m even, we have*

$$\frac{\prod_{1 \le i \le j \le n} (1 - y_i y_j t^2)}{\prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} = \sum_\lambda s_{[\lambda]}(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n) t^{|\lambda|}$$

and for m odd, we have

$$\frac{\prod_{1 \le i \le j \le n} (1 - y_i y_j t^2)}{\prod_{j=1}^n (1 - y_j t) \cdot \prod_{i=1}^n \prod_{j=1}^n (1 - x_i y_j t)(1 - x_i^{-1} y_j t)} = \sum_\lambda s_{[\lambda]}(x_1, \dots, x_n) s_\lambda(y_1, \dots, y_n) t^{|\lambda|}$$

where the sums are over all $\lambda_1 \ge \cdots \ge \lambda_n \ge 0$.

In Theorem 3.3.5, we obtained an identity of the form

$$s_\lambda(x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}) = s_{[\lambda]}(x_1, \dots, x_n) + \sum_{\mu,\ |\mu| < |\lambda|} m_{\lambda,\mu} s_{[\mu]}(x_1, \dots, x_n).$$

As in the symplectic case, this tells us there are certain integers $n_{\lambda,\mu}$ such that

$$s_{[\lambda]}(x_1, \dots, x_n) = \sum_\mu n_{\lambda,\mu} s_\mu(x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}).$$

Furthermore, $n_{\lambda,\lambda} = 1$ and $n_{\lambda,\mu} \ne 0$ implies $|\lambda| \ge |\mu|$. Also, since $m_{\lambda,\mu}$ is independent of n once $n \ge \ell(\lambda)$, the same is true for $n_{\lambda,\mu}$.

This leads to the following idea: in the ring of symmetric functions Λ, we define **orthogonal Schur functions** via

$$s_{[\lambda]} = \sum_\mu n_{\lambda,\mu} s_\mu.$$

By the upper-triangularity property, these also form a basis for Λ.

We define specialization maps

$$\begin{aligned} \pi_{\mathbf{O}(2n)} \colon \Lambda &\to \Lambda_{\mathbf{O}(m)} \\ s_{[\lambda]} &\mapsto \sum_\mu n_{\lambda,\mu} s_\mu(x_1, \dots, x_n, (1?), x_n^{-1}, \dots, x_1^{-1}). \end{aligned}$$

If $n \ge \ell(\lambda)$, then $\pi_{\mathbf{O}(m)}(s_{[\lambda]}) = s_{[\lambda]}(x_1, \dots, x_n)$, but otherwise we do not know. We can be more precise about what happens when $n < \ell(\lambda)$, but again I will postpone the discussion.

We consider the problem of multiplying two symplectic Schur functions. We have integers such that

$$s_{[\mu]} s_{[\nu]} = \sum_\lambda {}^{\mathbf{O}}c^\lambda_{\mu,\nu} s_{[\lambda]}$$

Theorem 3.4.6 (Newell–Littlewood product formula)**.** *We have*

$$s_{[\mu]}(x_1,\dots,x_n)s_{[\nu]}(x_1,\dots,x_n) = \sum_{\lambda,\alpha,\beta,\gamma} c^{\lambda}_{\beta,\gamma} c^{\mu}_{\beta,\alpha} c^{\nu}_{\gamma,\alpha} \pi_{\mathbf{O}(m)}(s_{[\lambda]})$$

and, in particular,

$$^{\mathbf{O}}c^{\lambda}_{\mu,\nu} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha,\beta} c^{\nu}_{\alpha,\gamma} c^{\lambda}_{\beta,\gamma}$$

where the sum is over all triples of partitions.

Surprisingly, we see that $^{\mathbf{O}}c^{\lambda}_{\mu,\nu} = {}^{\mathbf{Sp}}c^{\lambda}_{\mu,\nu}$.

4. Homogeneous spaces

4.1. **Projective space.** Given a vector space U, let $\mathbf{P}(U)$ denote the set of 1-dimensional subspaces in U. This is called **projective space**. If $U = \mathbf{C}^{n+1}$, we usually just write $\mathbf{P}^n$ in place of $\mathbf{P}(\mathbf{C}^{n+1})$ (the $n+1$ vs. n will be explained shortly). Alternatively, consider the action of $\mathbf{C}^*$ on U by scalar multiplication. This action is free on $U \setminus 0$ and $\mathbf{P}(U) = (U \setminus 0)/\mathbf{C}^*$. We use this description to give $\mathbf{P}(U)$ the quotient topology (where $U \setminus 0$ inherits the Zariski topology from U).

This gives a convenient set of coordinates for $\mathbf{P}^n$: we denote a point by $[a_0 : \cdots : a_n]$ with the understanding that at least one $a_i \neq 0$ and $[a_0 : \cdots : a_n] = [\lambda a_0 : \cdots : \lambda a_n]$ for any $\lambda \neq 0$. The topology on $\mathbf{P}^n$ can be described explicitly. An ideal $I \subset \mathbf{C}[x_0,\dots,x_n]$ is **homogeneous** if it is generated by homogeneous polynomials. Equivalently, this means that if $f \in I$ and $f = \sum_{i=0}^d f_i$ is its sum into degree i homogeneous polynomials, then each $f_i \in I$. Given a homogeneous polynomial f and $a = [a_0 : \cdots : a_n]$, it does not make sense to ask for the value $f(a)$, but it does make sense to ask if $f(a) = 0$.

So we define

$$Z(I) = \{a \in \mathbf{P}^n \mid f(a) = 0 \text{ for all } f \in I\}.$$

These are the closed subsets of the topology on $\mathbf{P}^n$, and $Z(I)$ is called a **projective variety**. Given a projective variety $X \subset \mathbf{P}^n$, we let $\mathbf{C}[X] = \mathbf{C}[x_0,\dots,x_n]/I_X$ where I_X is the ideal of all homogeneous polynomials f such that $f(a) = 0$ for all $a \in X$. Note that $\mathbf{C}[\mathbf{P}^n] = \mathbf{C}[x_0,\dots,x_n]$.

This is a special case of what we already discussed: given a projective subvariety $X \subset \mathbf{P}^n$, we let $\hat{X}$ denote its inverse image in $\mathbf{C}^{n+1} \setminus 0$ together with 0. This is the **affine cone** of X and is an affine variety. Then $\mathbf{C}[X] = \mathbf{C}[\hat{X}]$.

Remark 4.1.1. There is one subtle point: we haven't discussed isomorphisms between varieties in different ambient spaces, but it's important to note that the coordinate ring of an affine variety is independent of the embedding, while for a projective variety it does in general depend on the embedding. □

Finally, we justify the change in superscript from $n+1$ to n:

Proposition 4.1.2. $\dim \mathbf{P}^n = n$ *and* $\mathbf{P}^n$ *is irreducible.*

Proof. $\mathbf{GL}_{n+1}(\mathbf{C})$ acts transitively on $\mathbf{P}^n$ by $g \cdot \ell = g(\ell)$ where ℓ is a 1-dimensional subspace in $\mathbf{C}^{n+1}$. The stabilizer of the line spanned by e_1 consists of matrices g such that e_1 is an eigenvector of g, which just means the last n entries in the first column are 0. We see that this stabilizer has dimension $(n+1)^2 - n$, so $\dim \mathbf{P}^n = n$. □

Remark 4.1.3. We could also give $\mathbf{C}^{n+1}$ the standard Euclidean topology and then $\mathbf{P}^n$ inherits the Euclidean topology. This is convenient to do, though we won't really need to do so in this course. Note that vector spaces are compact in the Zariski topology (exercise) but not in the Euclidean topology. Projective space is compact in the Euclidean topology though. □

There's another description of $\mathbf{P}^n$ as a union of vector spaces. For each $i = 0, \dots, n$, let $U_i \subset \mathbf{P}^n$ denote the set of points a such that $a_i \neq 0$. We may as well assume $a_i = 1$ so that the remaining n coordinates on U_i are fixed numbers (there is no longer an equivalence relation). This shows that $U_i \cong \mathbf{C}^n$ and $\mathbf{P}^n = U_0 \cup \cdots \cup U_n$. Using the Euclidean topology, we can use this to define $\mathbf{P}^n$ as a complex manifold (the U_i are the charts).

The set of points where $a_n = 0$ is isomorphic to $\mathbf{P}^{n-1}$, so we get

$$\mathbf{P}^n = \mathbf{C}^n \amalg \mathbf{P}^{n-1}.$$

If we apply this decomposition recursively, we end up with

$$\mathbf{P}^n = \mathbf{C}^n \amalg \mathbf{C}^{n-1} \amalg \cdots \amalg \mathbf{C}^1 \amalg \mathbf{C}^0.$$

Explicitly, for $i < n$, $\mathbf{C}^i = \{a \in \mathbf{P}^n \mid a_{i+1} = \cdots = a_n = 0,\ a_i \neq 0\}$. Each $\mathbf{C}^i$ is closed under the action of the Borel subgroup of upper triangular matrices, and in fact is an orbit. This is an example of a Bruhat decomposition.

Remark 4.1.4. There are a lot of technical issues that need to be addressed to do a lot of basic constructions. For example, given projective varieties X, Y, can we make sense of the product $X \times Y$ as a projective variety? This embeds into a product of projective spaces, but note that $\mathbf{P}^n \times \mathbf{P}^m$ is not the same thing as $\mathbf{P}^{n+m}$! Another example: how to determine when two (embedded) varieties are isomorphic? i.e., can we give an intrinsic definition of varieties which does not rely on embedding them into projective space or vector spaces?

We will largely ignore these issues and work at a more intuitive (less rigorous) level. The arguments I make can be made rigorous, but one is better off learning the details in a course in algebraic geometry. □

4.2. **Grassmannians.** Our definition of projective space admits a natural definition: given a vector space U, let $\mathbf{Gr}(k, U)$ be the set of k-dimensional subspaces of U. Then $\mathbf{Gr}(1, U) = \mathbf{P}(U)$ and there is a natural identification $\mathbf{Gr}(k, U) = \mathbf{Gr}(\dim U - k, U^*)$. This is called a **Grassmannian**. For $U = \mathbf{C}^n$, we just write $\mathbf{Gr}(k, n)$ instead (so $\mathbf{P}^n = \mathbf{Gr}(1, n+1)$, which is slightly awkward for notation).

We can also realize this as a quotient: let $\mathbf{C}^{k \times n}$ be the space of $k \times n$ matrices. Given a subspace $W \subset U$, pick a basis for W and represent W as an element of $\mathbf{C}^{k \times n}$ by taking the matrix whose rows are the basis vectors just picked. This matrix is always full rank but the representative is not unique since a different choice of basis gives a different representative. However, any two choice of bases will differ by row operations, which corresponds to the action of $\mathbf{GL}_k(\mathbf{C})$ on the left. Letting $(\mathbf{C}^{k \times n})^\circ$ denote the full rank matrices, we have a natural identification

$$\mathbf{Gr}(k, n) = (\mathbf{C}^{k \times n})^\circ / \mathbf{GL}_k(\mathbf{C})$$

so we can again give it the quotient topology.

Here is another approach to define a topology on $\mathbf{Gr}(k, n)$ (we will omit the verification they are the same): given a k-dimensional subspace W, the exterior power $\bigwedge^k W$ is 1-dimensional and is a subspace of $\bigwedge^k U$. Explicitly, if $w_1, \dots, w_k$ is a basis for W, then $\bigwedge^k W$, as a subspace of $\bigwedge^k U$, is the span of $w_1 \wedge \cdots \wedge w_k$. Note that if we pick a different basis, then they differ by an element of $\mathbf{GL}_k(\mathbf{C})$, and $g(w_1) \wedge \cdots \wedge g(w_k) = (\det g) w_1 \wedge \cdots \wedge w_k$. Hence we get an injective (to recover W from $w_1 \wedge \cdots \wedge w_k$ just take the span of $w_1, \dots, w_k$) map

$$\mathbf{Gr}(k, U) \to \mathbf{P}(\bigwedge^k U)$$
$$W \mapsto \bigwedge^k W.$$

This is the **Plücker embedding**.

Even more explicitly, we have a basis for $\bigwedge^k \mathbf{C}^n$ indexed by k-element subsets of $\{1, \dots, n\}$, so we let the projective coordinates on $\mathbf{P}(\bigwedge^k \mathbf{C}^n)$ be labeled a_I for subsets I. Given a subspace W, we represent it as a $k \times n$ matrix as before. Let $f(W)_I$ be the determinant of the $k \times k$ submatrix with columns indexed by I. This gives a point $f(W) \in \mathbf{P}(\bigwedge^k \mathbf{C}^n)$ which is independent of the choice of basis. Then the coordinates $f(W)_I$ are called **Plücker coordinates**.

Proposition 4.2.1. $\mathbf{Gr}(k, n)$ *is a projective subvariety of* $\mathbf{P}(\bigwedge^k \mathbf{C}^n)$.

Proof. The image of $\mathbf{Gr}(k, n)$ consists of equivalence classes of lines of the form $w_1 \wedge \cdots \wedge w_k$, i.e., totally decomposable tensors. So it suffices to show there is a set of homogeneous polynomials f_i so that $f_i(\alpha) = 0$ if and only if $\alpha \in \mathbf{Gr}(k, n)$. Define a map

$$\varphi \colon \bigwedge^k \mathbf{C}^n \to \mathrm{Hom}(\mathbf{C}^n, \bigwedge^{k+1} \mathbf{C}^n)$$

(where $\mathrm{Hom}(A, B)$ denotes the vector space of linear maps $A \to B$) by $\varphi(\alpha)(u) = \alpha \wedge u$. Note that $\alpha \wedge u = 0$ if and only if $\alpha = \beta \wedge u$ for some $\beta \in \bigwedge^{k-1} \mathbf{C}^n$ (exercise). So if $\alpha = w_1 \wedge \cdots \wedge w_k$ is totally decomposable, then $\ker \varphi(\alpha) = \mathrm{span}(w_1, \dots, w_k)$ and hence $\mathrm{rank}\, \varphi(\alpha) = n - k$. Conversely, if $\ker \varphi(\alpha)$ is not totally decomposable, then $\dim \ker \varphi(\alpha) < k$. This means that α is totally decomposable if and only if all of the size $n - k + 1$ minors of $\varphi(\alpha)$ are 0. These are homogeneous polynomial functions of degree $n - k + 1$ (φ is linear) if we choose a basis for $\bigwedge^k \mathbf{C}^n$. $\square$

Remark 4.2.2. The functions we just produced do not generate the full ideal of polynomials that are 0 on $\mathbf{Gr}(k, n)$. In fact, the full ideal is generated by quadratic polynomials known as the Plücker equations. We won't need them, so we won't discuss it. $\square$

Example 4.2.3. Since $\mathbf{Gr}(k, n) \cong \mathbf{Gr}(n-k, n)$, the simplest example which isn't projective space is $k = 2$ and $n = 4$. In general, for $k = 2$, we can realize $\bigwedge^2 \mathbf{C}^n$ as the space of skew-symmetric matrices, and the Grassmannian $\mathbf{Gr}(2, n)$ consists of the lines spanned by elements of the form $v \wedge w$, which are rank 2 skew-symmetric matrices. Hence, we can describe it as the zero set of the 4×4 Pfaffians of principal submatrices. These are the Plücker equations in this case. When $n = 4$, there is just 1 Pfaffian, and the equation can be written as

$$x_{12}x_{34} - x_{13}x_{24} + x_{14}x_{23} = 0. \qquad \square$$

Proposition 4.2.4. $\dim \mathbf{Gr}(k, n) = k(n - k)$ *and* $\mathbf{Gr}(k, n)$ *is irreducible.*

Proof. $\mathbf{GL}_n(\mathbf{C})$ acts transitively on the set of k-dimensional subspaces. The stabilizer of the space spanned by $e_1, \dots, e_k$ consists of block matrices of the form $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$ where A is $k \times k$ and C is $(n-k) \times (n-k)$. The stabilizer has dimensions $n^2 - k(n-k)$, so $\dim \mathbf{Gr}(k,n) = k(n-k)$. □

4.2.1. *Bruhat decomposition.* We now describe a generalization of the Bruhat decomposition of projective space. Represent each k-dimensional subspace of $\mathbf{C}^n$ as the row span of a $k \times n$ matrix. Since it is full rank, at least one $k \times k$ submatrix is invertible. We index these submatrices by their columns $I = i_1 > \cdots > i_k$ and order these indices lexicographically. Let X_I be the set of subspaces such that the Ith submatrix has full rank, and the Jth submatrix does not have full rank for all $J > I$. Then $\mathbf{Gr}(k,n) = \amalg_I X_I$.

Given $W \in X_I$, there is a unique matrix whose row span is W so that the submatrix indexed by I is the matrix with 1's on the antidiagonal. In this form, all of the columns to the right of i_1 are 0, and more generally, all columns between i_j and i_{j-1} are in the span of the columns $i_{j-1}, \dots, i_1$, i.e., they are 0 in rows $\geq j$. Otherwise the entries can be chosen freely.

Example 4.2.5. If $k = 3$ and $n = 7$ and $I = (7,4,2)$, then a general element of X_I looks like

$$\begin{bmatrix} * & 0 & * & 0 & * & * & 1 \\ * & 0 & * & 1 & 0 & 0 & 0 \\ * & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$ can be any number. □

In the example, the free entries form a Young diagram. In general, this is true. We define $\lambda(I)$ by $\lambda(I)_j = i_j - (k+1-j)$. We see that $X_I \cong \mathbf{C}^{|\lambda(I)|}$. This indexing gives a bijection between the k-element subsets of $\{1, \dots, n\}$ and partitions λ satisfying $\ell(\lambda) \leq k$ and $\lambda_1 \leq n-k$. Furthermore, this normal form shows that each X_I is a single B-orbit.

As a variation of this idea, for each indexing set I, let U_I be the set of subspaces such that the submatrix indexed by I is invertible. Then there is a unique representative so that this is the antidiagonal matrix, but otherwise the rest of the $k(n-k)$ entries can be chosen freely, so $U_I \cong \mathbf{C}^{k(n-k)}$. Then $\mathbf{Gr}(k,n) = \bigcup_I U_I$. In the Euclidean topology, these are charts which can be used to give $\mathbf{Gr}(k,n)$ the structure of a complex manifold.

4.2.2. *Vector bundles.* Roughly speaking, a rank n vector bundle over a variety X will be a space Y with a map $\pi \colon Y \to X$ such that the fibers $Y_x = \pi^{-1}(x)$ are n-dimensional vector spaces and there is an open covering $\bigcup U_i$ of X so that $\pi^{-1}(U_i) \cong \mathbf{C}^n \times U_i$.

Consider the product $\mathbf{C}^n \times \mathbf{Gr}(k,n)$. We will think of this as a trivial vector bundle over $\mathbf{Gr}(k,n)$. Define

$$\mathcal{R} = \{(v, W) \in \mathbf{C}^n \times \mathbf{Gr}(k,n) \mid v \in W\}.$$

This has a map $\pi \colon \mathcal{R} \to \mathbf{Gr}(k,n)$ given by $\pi(v,W) = W$ and $\pi^{-1}(W)$ is naturally identified with W itself. This is the **tautological subbundle** on $\mathbf{Gr}(k,n)$. It is a vector subbundle of the trivial bundle. We let $\mathcal{Q}$ denote the quotient bundle $\mathbf{C}^n \times \mathbf{Gr}(k,n)/\mathcal{R}$ (roughly: quotient bundles do nothing to the base space X but the fibers get replaced by the appropriate quotient spaces), so that the fiber of $\mathcal{Q}$ over W is the quotient space $\mathbf{C}^n/W$. Given two vector bundles E and F on a variety X, $\mathrm{Hom}(E,F)$ is the vector bundle on X whose fibers are $\mathrm{Hom}(E,F)_x = \mathrm{Hom}(E_x, F_x)$. We can import essentially all of the operations on vector spaces to vector bundles: direct sums, tensor products, symmetric and exterior powers, Schur functors, etc.

Given a vector bundle E over X, a section $s\colon X \to E$ is a continuous function such that $\pi \circ s$ is the identity on X. There is a canonical zero section s_0 which sends x to $0 \in E_x$. We let $X_0 = s_0(X)$, which is a closed subset of E.

4.3. **(Partial) flag varieties.** Given $\mathbf{d} = d_1 < d_2 < \cdots < d_r < n$, let $\mathbf{Fl}(\mathbf{d}; n) = \mathbf{Fl}(d_1, d_2, \dots, d_r; n)$ denote the set of r-tuples of subspaces $(W_{d_1}, W_{d_2}, \dots, W_{d_r})$ such that $W_{d_1} \subset W_{d_2} \subset \cdots \subset W_{d_r}$ and $\dim W_{d_i} = d_i$. This is a **partial flag variety** and is naturally a subset of $\mathbf{Gr}(d_1, n) \times \mathbf{Gr}(d_2, n) \times \cdots \times \mathbf{Gr}(d_2, n)$. The **(full) flag variety** $\mathbf{Fl}(n)$ is the case when $r = n-1$ and $\mathbf{d} = 1 < 2 < \cdots < n-1$. In the latter case, the points are called complete flags.

Proposition 4.3.1. *$\mathbf{Fl}(\mathbf{d}; n)$ is a closed subset of $\mathbf{Gr}(d_1, n) \times \cdots \times \mathbf{Gr}(d_r, n)$ and hence is a projective variety.*

Proof. We need to show that the condition $W_{d_i} \subset W_{d_{i+1}}$ defines a closed subset. Let $X = \mathbf{Gr}(d_1, n) \times \cdots \times \mathbf{Gr}(d_r, n)$ and define $\mathcal{R}_{d_i} \subset \mathbf{C}^n \times X$ as tuples $(v, W_{d_1}, \dots, W_{d_r})$ such that $v \in W_{d_i}$ and let $\mathcal{Q}_{d_i} = (\mathbf{C}^n \times X)/\mathcal{R}_{d_i}$. Consider the composition $\mathcal{R}_{d_i} \to \mathcal{Q}_{d_{i+1}}$. This map gives a map $s\colon X \to \mathrm{Hom}(\mathcal{R}_{d_i}, \mathcal{Q}_{d_{i+1}})$ and $s^{-1}(X_0)$ consists of subspaces such that $W_{d_i} \subset W_{d_{i+1}}$, so the condition is closed. Then $\mathbf{Fl}(\mathbf{d}; n)$ is the intersection of $r-1$ of these closed subsets, so is itself closed. $\square$

$\mathbf{GL}_n(\mathbf{C})$ acts transitively on any partial flag variety. Let's consider the stabilizer subgroup of the standard flag in $\mathbf{Fl}(\mathbf{d}; n)$, i.e., W_{d_i} is spanned by $e_1, \dots, e_{d_i}$. We see that it consists of block (r blocks) upper-triangular matrices where the kth block has size d_k. This proves the following:

Proposition 4.3.2. $\dim \mathbf{Fl}(\mathbf{d}; n) = \sum_{i=1}^r (d_i - d_{i-1})(n - d_i)$ *($d_0 = 0$) and $\mathbf{Fl}(\mathbf{d}; n)$ is irreducible. In particular,* $\dim \mathbf{Fl}(n) = \binom{n}{2}$.

There is another way to compute this dimension using the following fact:

If $f\colon X \to Y$ is a polynomial map between irreducible varieties such that $\dim f^{-1}(y) = e$ for all $y \in Y$, then $\dim X = \dim Y + e$.

Note that we have forgetful maps $\mathbf{Fl}(\mathbf{d}; n) \to \mathbf{Fl}(\mathbf{d}'; n)$ where $\mathbf{d}'$ is any subsequence of $\mathbf{d}$. The fibers can be identified with flag varieties of a certain type (we leave this as an exercise) and the dimension can be computed in this way.

Going back to stabilizers, we see that the stabilizer subgroup of the standard flag in $\mathbf{Fl}(n)$ is the Borel subgroup of upper triangular matrices. In fact, given any complete flag $W_1 \subset W_2 \subset \cdots$, we get a basis $w_1, w_2, \dots, w_n$ by taking $w_i \in W_i \setminus W_{i-1}$ (and $w_1 \in W_1 \setminus 0$) and the corresponding Borel subgroup is the stabilizer of $W_\bullet$. We see then that the data of a Borel subgroup only depends on a complete flag, which is less information than a basis. This actually gives us a bijection between Borel subgroups and complete flags, so $\mathbf{Fl}(n)$ is also known as the variety of Borel subgroups. It is also denoted by $\mathbf{GL}_n(\mathbf{C})/B$.

Any subgroup of $\mathbf{GL}_n(\mathbf{C})$ that is closed in the Zariski topology and contains a Borel subgroup is known as a **parabolic subgroup**. We see that the stabilizers of partial flags are always parabolic subgroups. In fact, every parabolic subgroup is the stabilizer of some partial flag (we don't prove this). Parabolic subgroups are not in general conjugate; they have a type which in this case corresponds to the discrete datum $\mathbf{d}$. But parabolic subgroups of the same type $\mathbf{d}$ are conjugate to one another (since $\mathbf{GL}_n(\mathbf{C})$ acts transitively on $\mathbf{Fl}(\mathbf{d}; n)$). Hence $\mathbf{Fl}(\mathbf{d}; n)$ is usually denoted by something like $\mathbf{GL}_n(\mathbf{C})/P_{\mathbf{d}}$.

More generally, the partial flag varieties are known as **homogeneous spaces**: these are projective varieties with a transitive action of our group. It turns out that we have described all of them for $\mathbf{GL}_n(\mathbf{C})$ (there is also the example of a single point, but we will exclude it).

Here is another realization of partial flag varieties. Consider an irreducible representation $\mathbf{S}_\lambda(\mathbf{C}^n)$ and the corresponding projective space $\mathbf{P}(\mathbf{S}_\lambda(\mathbf{C}^n))$. Define

$$X_\lambda = \{[x] \in \mathbf{P}(\mathbf{S}_\lambda \mathbf{C}^n) \mid x \text{ is a highest weight vector for some Borel subgroup}\}.$$

Lemma 4.3.3. *X_λ is closed under the action of $\mathbf{GL}_n\mathbf{C}$ and is in fact a single orbit.*

Proof. Let v be a highest weight vector for a Borel subgroup B. Then gv is a highest weight vector for $g^{-1}Bg$, so X_λ is closed under $\mathbf{GL}_n\mathbf{C}$.

Let w be another highest weight vector with respect to another Borel subgroup B'. We know that any two Borel subgroups are conjugate, so $B' = gBg^{-1}$ for some $g \in \mathbf{GL}_n(\mathbf{C})$. This means that $gbg^{-1}[w] = [w]$ for all $b \in B$, or equivalently that $bg^{-1}[w] = g^{-1}[w]$ for all $b \in B$, i.e., that $g^{-1}w$ is a highest weight vector for B. By uniqueness of highest weight vectors, this means that $g^{-1}[w] = [v]$. $\square$

Let us compute the stabilizer of the highest weight vector with respect to the standard Borel subgroup of upper triangular matrices. First, consider $\lambda = (1^d)$, so $\mathbf{S}_\lambda \mathbf{C}^n = \bigwedge^d \mathbf{C}^n$. The highest weight vector is any multiple of $e_1 \wedge \cdots \wedge e_d$ and the stabilizer is the group of block upper-triangular matrices with block sizes d and $n-d$. This is the parabolic subgroup P_d.

Recall that $\mathbf{S}_\lambda \mathbf{C}^n$ can be constructed as a quotient of $\bigwedge^{\mu_1} \mathbf{C}^n \otimes \cdots \otimes \bigwedge^{\mu_s} \mathbf{C}^n$ where $\mu = \lambda^\dagger$. The highest weight vector of $\mathbf{S}_\lambda \mathbf{C}^n$ is the image of the tensor product of the corresponding highest weight vectors of each exterior power. So we see that the stabilizer is $P_{\mu_1} \cap \cdots \cap P_{\mu_r}$ which is the stabilizer of the standard flag consisting of subspaces of dimensions $\{\mu_1, \dots, \mu_r\}$ (note that multiplicities among these numbers do not change the stabilizer). In particular:

Proposition 4.3.4. *$X_\lambda \cong \mathbf{Fl}(\mathbf{d}; n)$ where $\mathbf{d}$ is obtained from $\lambda^\dagger$ by removing redundancies and sorting in increasing order.*

4.4. **Borel–Weil theorem.** Let $G = \mathbf{GL}_n(\mathbf{C})$, $P = P_{\mathbf{d}}$, $X = \mathbf{Fl}(\mathbf{d}; n)$ (what follows will generalize to other groups).

A vector bundle E over the homogeneous space X is called **homogeneous** if E has an (algebraic) action of G such that $\pi(ge) = g\pi(e)$ for all $g \in G$ and $e \in E$. This implies that the fiber E_x is a representation of the stabilizer of x. For concreteness if x is the standard flag, then E_x is a representation of $P_{\mathbf{d}}$.

Theorem 4.4.1. *E is completely encoded by the representation E_x. More formally, there is an equivalence between the category of homogeneous bundles on G/P and the category of rational P-representations.*

We have explained how to go from a homogeneous bundle to a P-representation. In the other direction, if V is a P-representation, then we define

$$\mathrm{Ind}_P^G V := (G \times V)/\sim, \qquad (g, v) \sim (gp^{-1}, pv) \text{ for } g \in G,\, p \in P,\, v \in V.$$

We define $\pi \colon \mathrm{Ind}_P^G V \to G/P$ by $\pi(g, v) = gP$. There is an action of G on $\mathrm{Ind}_P^G V$ by $g' \cdot (g, v) = (g'g, v)$ which is compatible with π. Then $\pi^{-1}(gP)$ is naturally identified with

V, so we see that $\mathrm{Ind}_P^G V$ is a homogeneous bundle whose rank is $\dim V$. Also, $\pi^{-1}(P)$ is the original representation we started with. We omit the discussion that $\mathrm{Ind}_P^G E_x = E$.

Given a vector bundle E on X, we let $\mathrm{H}^0(X; E)$ denote the space of sections, i.e., $s \colon X \to E$ such that $\pi \circ s = \mathrm{id}_X$. This is a complex vector space since the sum of two sections makes sense as does scalar multiplication. A basic fact from algebraic geometry is that if X is a projective variety, then $\mathrm{H}^0(X; E)$ is finite-dimensional. If E is homogeneous, then it is is also a rational representation of G via $(g \cdot s)(x) = g(s(g^{-1}x))$.

We have that $T \cong B/[B, B]$ and in particular, we have a surjection $B \to T$. Given a weight λ, we have a 1-dimensional rational representation $\mathbf{C}_\lambda$ of T defined by $t \cdot z = \lambda(t)z$. Via the surjection $B \to T$, this gives us a rational representation of B that we also call $\mathbf{C}_\lambda$. In particular,

$$\mathcal{L}_\lambda = \mathrm{Ind}_B^G \mathbf{C}_\lambda$$

is a vector bundle of rank 1 (i.e., a **line bundle**) on G/B.

Theorem 4.4.2 (Borel–Weil theorem)**.** *If λ is dominant (i.e., $\lambda_1 \ge \cdots \ge \lambda_n$), then $\mathrm{H}^0(G/B; \mathcal{L}_\lambda)^* \cong \mathbf{S}_\lambda \mathbf{C}^n$.*

Proof. Let L be the subgroup of lower-triangular matrices with 1's on the diagonal. The number of highest weight vectors (up to scalar multiple) in $\mathrm{H}^0(G/B; \mathcal{L}_\lambda)^*$ is the same as the number of L-fixed vectors in $\mathrm{H}^0(G/B; \mathcal{L}_\lambda)$.

In Lemma 1.2.9, we showed that the set of matrices of the form ℓb with $\ell \in L$ and $b \in B$ is open and dense in the space of $n \times n$ matrices (and hence also in $\mathbf{GL}_n(\mathbf{C})$). This also implies that $\{\ell B \mid \ell \in L\}$ is dense in G/B. If $s \in \mathrm{H}^0(G/B; \mathcal{L}_\lambda)$ is an L-fixed vector, then $\ell(s(\ell^{-1}B)) = (\ell \cdot s)(B) = s(B)$ for all $\ell \in L$, and hence s is determined by its value on B since this determines its values on a dense subset. This proves that L-fixed vectors are unique up to scalar multiple.

Now we construct an L-fixed section. Note that a section of $\mathcal{L}_\lambda = (G \times \mathbf{C}_\lambda)/\sim$ can be written as $s(gB) = (g, f(g))$ where $f \colon G \to \mathbf{C}$ is a function such that $f(gb) = \lambda(b)f(g)$ for all $b \in B$ and $g \in G$. We define such a function as follows. Let $f_i(g)$ be the determinant of the upper left $i \times i$ submatrix of g. Then define $f(g) = \prod_{i=1}^n f_i(g)^{\lambda_i - \lambda_{i+1}}$ where we set $\lambda_{n+1} = 0$. Then $f(gb) = f(g)f(b) = \lambda(b)f(g)$, so we have a section. If t is diagonal with entries $x_1, \dots, x_n$, then

$$(t \cdot s)(gB) = t(s(t^{-1}gB)) = t(t^{-1}g, f(t^{-1})f(g)) = t(t^{-1}g, \lambda(t)^{-1}f(g)) = (g, \lambda(t)^{-1}f(g)).$$

Then s has weight $-\lambda$, and so the corresponding dual highest weight vector in $\mathrm{H}^0(G/B; \mathcal{L}_\lambda)^*$ has weight λ. ☐

Remark 4.4.3. Since $\mathbf{C}_\lambda \otimes \mathbf{C}_\mu = \mathbf{C}_{\lambda+\mu}$ as representations of T (and hence B), it follows that $\mathcal{L}_\lambda \otimes \mathcal{L}_\mu \cong \mathcal{L}_{\lambda+\mu}$ as homogeneous bundles. In general, the set of line bundles (modulo isomorphism) on a space is a group under tensor product, called the **Picard group** (inverse is given by taking duals). Note that the line bundle $\mathcal{L}_{(1^n)}$ has a section which is nowhere zero: $s(gB) = (g, \det g)$. We use this to define an isomorphism $G/B \times \mathbf{C} \to \mathcal{L}_{(1^n)}$ by $(gB, \alpha) \mapsto (g, \alpha \det g)/\sim$. Hence $\mathcal{L}_\lambda \cong \mathcal{L}_\lambda \otimes \mathcal{L}_{(d^n)}$ for any $d \in \mathbf{Z}$ as line bundles (but they are non-isomorphic as homogeneous bundles). It turns out these are all of the redundancies, so the Picard group of G/B is isomorphic to $\mathbf{Z}^{n-1}$. ☐

Remark 4.4.4. All of this can be done in a relative setting. Instead of taking the flag variety of a vector space, we can make sense of the flag variety of a vector bundle. Let E be a vector bundle of rank n on X. Intuitively, $\mathbf{Fl}(E)$ is a space together with a map $\pi\colon \mathbf{Fl}(E) \to X$ so that $\pi^{-1}(x)$ is naturally identified with $\mathbf{Fl}(E|_x)$. Given a weight $\lambda = (\lambda_1, \dots, \lambda_n)$, we can make sense of the line bundle $\mathcal{L}_\lambda$ over $\mathbf{Fl}(E)$. The analogue of taking sections is to compute the *pushforward* $\pi_* \mathcal{L}_\lambda$, which will end up being the vector bundle $(\mathbf{S}_\lambda E)^*$ on X. Similarly, we can construct relative partial flag varieties. $\square$

Remark 4.4.5. You may wonder why H^0 is used for the notation of sections. It turns out that H^0 is a left-exact functor: given a subbundle $E \subset F$, we have maps $\mathrm{H}^0(X; E) \to \mathrm{H}^0(X; F) \to \mathrm{H}^0(X; F/E)$ whose composition is 0. Left-exact means that the first map is injective and that the image of the first map equals the kernel of the second map. In general, the second map need not be surjective. Homological algebra tells us that there is a sequence of spaces $\mathrm{H}^i(X; E)$ for $i \geq 0$ that repair this situation in some sense. Namely, we can extend the sequence to

$$\cdots \to \mathrm{H}^0(X; F/E) \to \mathrm{H}^1(X; E) \to \mathrm{H}^1(X; F) \to \cdots$$

so that at all stages, the image of each map is equal to the kernel of the next map. *Bott's theorem* (also called Borel–Weil–Bott) explains how to compute $\mathrm{H}^i(G/B; \mathcal{L}_\lambda)$ in general. We won't discuss it, but only mention that for each λ, there is at most one i so that the space is nonzero. $\square$

4.5. **Symplectic and orthogonal analogues.** The homogeneous spaces for the symplectic group are an easy variation of flag varieties. Let $\mathbf{C}^{2n}$ be a symplectic vector space with symplectic form ω and pick integers $\mathbf{d} = (d_1, \dots, d_r)$ satisfying $0 < d_1 < \cdots < d_r \leq n$. The **isotropic (partial) flag variety** $\mathbf{IFl}(\mathbf{d}; \mathbf{C}^{2n})$, or just $\mathbf{IFl}(\mathbf{d}; 2n)$, is the set of partial flags $W_{d_1} \subset \cdots \subset W_{d_r} \subset \mathbf{C}^{2n}$ such that $\dim W_{d_j} = d_j$ and each W_{d_j} is an isotropic subspace. We leave it as an exercise to see that $\mathbf{Sp}_{2n}(\mathbf{C})$ acts transitively on $\mathbf{IFl}(\mathbf{d}; 2n)$. We let $\mathbf{IFl}(2n)$ denote the full isotropic flag variety, i.e., the case when $\mathbf{d} = (1, 2, \dots, n)$.

Proposition 4.5.1. $\mathbf{IFl}(\mathbf{d}; 2n)$ *is a projective variety.*

Proof. Consider $\mathbf{Fl}(\mathbf{d}; 2n)$. Then $\mathbf{IFl}(\mathbf{d}; 2n)$ is the subset where W_{d_r} is isotropic. There is a tautological subbundle $\mathcal{R}_{d_r} \subset \mathbf{C}^{2n} \times \mathbf{Fl}(\mathbf{d}; 2n)$ given by $(v, W_\bullet)$ such that $v \in W_{d_r}$. We define a section s of $(\bigwedge^2 \mathcal{R}_{d_r})^*$ as follows. Given a flag $W_\bullet$, the fiber is $\bigwedge^2 W_{d_r}^*$. We define an element $s(W_\bullet)$ which is a linear functional on $\bigwedge^2 W_{d_r}$ defined by $u \wedge v \mapsto \omega(u, v)$. Then $\mathbf{IFl}(\mathbf{d}; 2n) = s^{-1}(\mathbf{IFl}(\mathbf{d}; 2n)_0)$, so is closed. $\square$

Remark 4.5.2. If E is a vector bundle on a space X, then we call it symplectic if it there is a linear map $\bigwedge^2 E \to \mathbf{C} \times X$, where the latter is the trivial line bundle, such that the restriction of this to each fiber is a symplectic form. Using this, we can construct a relative isotropic flag variety $\mathbf{IFl}(\mathbf{d}; E)$ which has a map π to X such that $\pi^{-1}(x) = \mathbf{IFl}(\mathbf{d}; E_x)$. More generally, we could replace $\mathbf{C} \times X$ by any line bundle on X. $\square$

If $\mathbf{d} = (1)$, then $\mathbf{IFl}(\mathbf{d}; 2n) = \mathbf{P}^{2n-1}$ since every line is automatically isotropic.

Proposition 4.5.3. $\dim \mathbf{IFl}(2n) = n^2$.

Proof. Let $\mathcal{R}$ be the tautological subbundle for $\mathbf{P}^{2n-1}$. Then define $\mathcal{R}^\perp \subset \mathbf{C}^{2n} \times \mathbf{P}^{2n-1}$ as (v, W) such that $\omega(v, u) = 0$ for all $u \in W$. Then $\mathcal{R} \subset \mathcal{R}^\perp$ and $\mathcal{R}^\perp/\mathcal{R}$ is a symplectic bundle. Furthermore, $\mathbf{IFl}(2n) = \mathbf{IFl}(\mathcal{R}^\perp/\mathcal{R})$ and the fibers are all isomorphic to $\mathbf{IFl}(2n-2)$, so

$$\dim \mathbf{IFl}(2n) = \dim \mathbf{P}^{2n-1} + \dim \mathbf{IFl}(2n-2).$$

By induction, we see that $\dim \mathbf{IFl}(2n) = n^2$. $\square$

There is a surjective map $\mathbf{IFl}(2n) \to \mathbf{IFl}(\mathbf{d}; 2n)$ by forgetting the subspaces that are not indexed by $\mathbf{d}$. The fibers are products of flag varieties (for the general linear group) of the quotient spaces $W_{d_i}/W_{d_{i-1}}$, and this can be used to compute $\dim \mathbf{Fl}(\mathbf{d}; 2n)$ for any $\mathbf{d}$.

The homogeneous spaces for the special orthogonal group are also an easy variation of flag varieties though there are a few minor technical subtleties in the even case. Let $\mathbf{C}^{2n+1}$ be an odd-dimensional orthogonal vector space with orthogonal form ω and pick integers $\mathbf{d} = (d_1, \dots, d_r)$ satisfying $0 < d_1 < \cdots < d_r \le n$. The **isotropic (partial) flag variety** $\mathbf{OFl}(\mathbf{d}; \mathbf{C}^{2n+1})$, or just $\mathbf{OFl}(\mathbf{d}; 2n+1)$, is the set of partial flags $W_{d_1} \subset \cdots \subset W_{d_r} \subset \mathbf{C}^{2n+1}$ such that $\dim W_{d_j} = d_j$ and each W_{d_j} is an isotropic subspace. We leave it as an exercise to see that $\mathbf{SO}_{2n+1}(\mathbf{C})$ acts transitively on $\mathbf{OFl}(\mathbf{d}; 2n+1)$.

It is tempting to make the same definition in the even-dimensional case. Recall that there are 2 orbits of n-dimensional isotropic subspaces under $\mathbf{SO}_{2n}(\mathbf{C})$. We can distinguish them based on the parity of $\dim(W \cap \operatorname{span}\{e_1, \dots, e_n\})$. We call W even or odd based on this parity. This suggests that the datum $\mathbf{d}$ should allow for n and n', where n' will denote an odd n-dimensional isotropic subspace. However, the following result suggests that a further change should be made.

Lemma 4.5.4. *Given an $(n-1)$-dimensional isotropic subspace of an orthogonal space $\mathbf{C}^{2n}$, there are exactly two n-dimensional isotropic subspaces that contain it.*

Proof. Let $W \subset \mathbf{C}^{2n}$ be an $(n-1)$-dimensional isotropic subspace. Then $W^\perp/W$ is a 2-dimensional orthogonal space and 1-dimensional subspaces of $W^\perp/W$ are in bijection with n-dimensional isotropic subspaces that contain W. But the orthogonal form can be written as xy after some change of coordinates, so there are exactly 2 solutions up to scalar multiple. $\square$

This implies that once we include $n-1$ in $\mathbf{d}$, the odd and even n-dimensional subspaces are determined, i.e., including $n-1$ in $\mathbf{d}$ makes n and n' redundant. There are a few choices to remove these redundancies, but we stick to the following. Consider the set $\{1, 2, \dots, n-2, n, n'\}$ and let $\mathbf{d}$ be a subset. The **isotropic (partial) flag variety** $\mathbf{OFl}(\mathbf{d}; 2n)$ is the set of subspaces $\{W_i \mid i \in \mathbf{d}\}$ such that:

- If $i < j$ then $W_i \subset W_j$ (we use the convention $i < n'$ if $i < n$)
- If $n, n' \in \mathbf{d}$, then $\dim(W_n \cap W_{n'}) = n - 1$.

Note that if $n, n' \in \mathbf{d}$, then we could replace both by $n-1$ thanks to the lemma above. However, it is then awkward to describe the cases where only one of n and n' belongs to $\mathbf{d}$.

In both cases, we let $\mathbf{OFl}(m)$ denote the full isotropic flag variety, i.e., when $\mathbf{d}$ contains all possible elements.

Proposition 4.5.5. $\mathbf{OFl}(\mathbf{d}; m)$ *is a projective variety.*

The proof is basically the same as for the symplectic case. The only thing to note is when $m = 2n$ and both $n, n' \in \mathbf{d}$. In that case, we should replace them with $n-1$ so that we can embed in the appropriate flag variety.

Remark 4.5.6. If E is a vector bundle on a space X, then we call it orthogonal if there is a linear map $\operatorname{Sym}^2 E \to \mathbf{C} \times X$, where the latter is the trivial line bundle, such that the restriction of this to each fiber is an orthogonal form. Using this, we can construct a relative isotropic flag variety $\mathbf{OFl}(\mathbf{d}; E)$ which has a map π to X such that $\pi^{-1}(x) = \mathbf{OFl}(\mathbf{d}; E_x)$. More generally, we could replace $\mathbf{C} \times X$ by any line bundle on X. $\square$

If $\mathbf{d} = (1)$, then $\mathbf{OFl}(\mathbf{d}; m)$ is defined by a single quadratic polynomial $\sum_{i=1}^{m-n} x_i x_{m+1-i} = 0$ in $\mathbf{P}^{m-1}$, so it has dimension $m-2$.

Proposition 4.5.7. $\dim \mathbf{OFl}(m) = n(m-n-1)$.

Proof. Let $\mathcal{R}$ be the tautological subbundle for $\mathbf{OFl}(1; m)$. Then $\mathcal{R}^\perp/\mathcal{R}$ is an orthogonal bundle and $\mathbf{OFl}(m) = \mathbf{OFl}(\mathcal{R}^\perp/\mathcal{R})$ and the fibers are all isomorphic to $\mathbf{OFl}(m-2)$, so

$$\dim \mathbf{OFl}(m) = m - 2 + \dim \mathbf{OFl}(m-2).$$

The base cases are $m = 2$, in which case $\mathbf{OFl}(2)$ is 2 points, and hence 0-dimensional, and $m = 3$, in which case $\mathbf{OFl}(3)$ is a curve in $\mathbf{P}^2$, and hence 1-dimensional. So by induction, we see that $\dim \mathbf{OFl}(m) = n(m-n-1)$. $\square$

As in the symplectic case, we can use this to compute the dimensions of any $\mathbf{OFl}(\mathbf{d}; m)$.

An analogue of the Borel–Weil theorem can be setup in both cases. The details are essentially the same, so we omit the discussion.

4.6. **Kempf–Weyman collapsing.** Let U be a vector space and consider the trivial vector bundle $U \times X$ over a projective variety X. Given a subbundle $\mathcal{S} \subset U \times X$, we can consider the first projection map $\mathcal{S} \to U$. By general theory, since X is projective, the image Y of $\mathcal{S}$ in U is closed. We call it a **collapsing** of the vector bundle $\mathcal{S}$. If X is irreducible, then the same is true for $\mathcal{S}$, and so Y is also automatically irreducible. In some favorable cases, we can use this to compute the coordinate ring of Y, generators for its ideal, and more. However, this would requires a lot more technical background than I wish to cover in this course, so we will just focus on examples, starting with some that we've already studied.

Example 4.6.1 (Determinantal varieties)**.** Let E, F be vector spaces and consider $U = \mathrm{Hom}(E, F)$, the space of linear maps $E \to F$. Assume that $\dim E \leq \dim F$ and fix an integer $r < \dim E$. Let Y be the subset of U consisting of linear maps of rank $\leq r$. Note that Y is an affine subvariety (it is the zero locus of the determinants of the submatrices of size $r+1$ once we pick bases).

Note that Y is not linear in the sense that it is not closed under addition. However, we can "linearize" Y as follows. For a fixed r-dimensional subspace $W \subset F$, the set of arbitrary maps $E \to W$ is a linear subspace of Y. If we vary the choice of W (alternatively, consider the closure of this linear subspace under the action of $\mathbf{GL}(F)$), then we get Y.

This suggests that we can realize Y as a collapsed vector bundle. Let $X = \mathbf{Gr}(r, F)$, and let $\mathcal{R}$ be the tautological subbundle on X. Consider $\mathcal{S} = \mathrm{Hom}(E, \mathcal{R}) \cong E^* \otimes \mathcal{R} \subset (E^* \otimes F) \times X = U \times X$. Then the image of $\mathcal{S}$ in U is exactly Y.

More explicitly, $\mathcal{S} = \{(f, W) \mid W \in \mathbf{Gr}(r, F),\ f \colon E \to W\}$ and the map π is $\pi(f, W) = f$. Then $\pi^{-1}(f) = \{(f, W) \mid \mathrm{image}\, f \subseteq W\}$. If $\mathrm{rank}\, f = r$, then the fiber is exactly one point and π is *birational isomorphism*, i.e., is an isomorphism away from a closed subset (technically, we have to also show that the inverse $f \mapsto (f, \mathrm{image}\, f)$ is an algebraic function). In fact, the Grassmannian is a smooth projective variety (automatic since it is homogeneous: a general point is smooth and all points are the same) and so $\mathcal{S}$ is smooth as well, which means that it is a *desingularization* of Y.

The varieties of symmetric and skew-symmetric matrices of bounded rank also fits into this picture. $\square$

Example 4.6.2 (Isotropic maps)**.** Let E be a vector space of dimension m and V a symplectic space of dimension $2n$. Again let $U = \mathrm{Hom}(E, V)$ and let Y be the set of maps

$f\colon E \to V$ such that $f(E) \subset V$ is an isotropic space. This behaves much like the previous example: let $X = \mathbf{IGr}(\min(m,n), V)$. Again, let $\mathcal{R}$ be the tautological subbundle of $V \times X$. We take $\mathcal{S} = \mathrm{Hom}(E, \mathcal{R})$. All of the above discussion applies here. We can also modify Y by considering only maps such that $f(E)$ is isotropic and $\operatorname{rank} f \le r$ for some fixed $r \le \min(m,n)$.

We can also take V to be an orthogonal space instead if we like. There is one caveat: if $\dim V = 2n$ is even, and $m \ge n$, then we need to take $X = \mathbf{OGr}(n, V) \amalg \mathbf{OGr}(n', V)$. $\square$

Example 4.6.3 (Nilpotent cone)**.** Let E be a vector space of dimension n and let $U = \mathrm{Hom}(E, E)$. A matrix $x \in U$ is **nilpotent** if some power of x is 0. In fact, this forces $x^n = 0$, so the set of Y of nilpotent matrices is an affine subvariety called the **nilpotent cone**. It is the zero locus of the entries of x^n for example. A more efficient set of equations is to take the coefficients of the characteristic polynomial of x: nilpotent is equivalent to having characteristic polynomial equal to t^n, so expanding $\det(x - tI_n)$ gives polynomials in the entries of x that are 0.

To realize this as a collapsed vector bundle, we try to find a linear subspace inside of Y. Here's one: if we pick a basis, then a quick computation shows that all matrices which are strictly upper-triangular with respect to this basis are automatically nilpotent. Actually, we just need a choice of complete flag rather than a choice of basis. This is a linear subspace. Furthermore, we claim that closing this subspace by the action of $\mathbf{GL}(E)$ (change of basis) gives all of Y. Given a nilpotent matrix, we have a sequence of subspaces $\operatorname{image} x \supset \operatorname{image} x^2 \supset \cdots \supset \operatorname{image} x^{n-1}$. If $\operatorname{image} x^i \ne 0$, then it strictly contains $\operatorname{image} x^{i+1}$ (if not, then x would not be nilpotent). Hence we get a partial flag. If we refine this to a complete flag $F_1 \subset F_2 \subset \cdots \subset F_{n-1}$, then $x(F_i) \subseteq F_{i-1}$ for all i. There is an element of $\mathbf{GL}(E)$ that takes the standard complete flag to this complete flag, so our claim is proven.

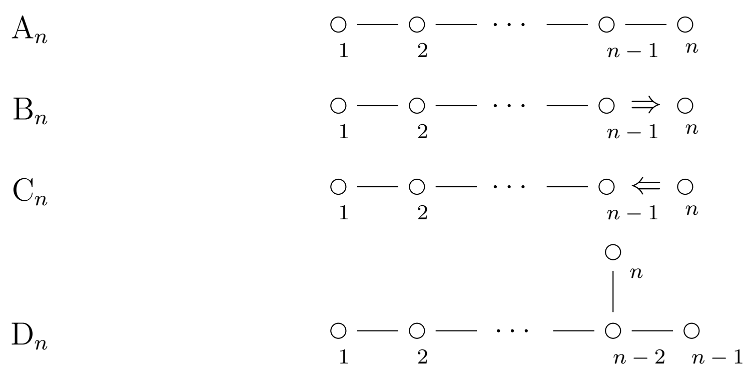
Take $X = \mathbf{Fl}(E)$ and let $\mathcal{R}_i$ be the ith tautological subbundle on X. Take $\mathcal{S}$ to be the subbundle of $U \times X$ consisting of pairs $(x, F_\bullet)$ such that $x(F_i) \subseteq F_{i-1}$ for all i. (Remark: $\mathcal{S}$ is the cotangent bundle of X.) Then the image of $\mathcal{S}$ in U is Y. A general choice of $x \in Y$ has one Jordan block, so that $\operatorname{image} x \supset \operatorname{image} x^2 \supset \cdots \supset \operatorname{image} x^{n-1}$ is already a complete flag. In that case, we see that $\pi^{-1}(x)$ is a single point and so $\pi\colon \mathcal{S} \to Y$ is a birational isomorphism. This implies that $\dim Y = \dim X + \operatorname{rank} \mathcal{S}$. We leave it as an exercise to show that the ideal of Y is generated by the coefficients of the characteristic polynomial.

Rather than deal with all nilpotent matrices, we can stratify Y by the sizes of the Jordan blocks in Jordan normal form. We leave this as an exercise. $\square$

Example 4.6.4 (Binary forms)**.** Let $U = \mathrm{Sym}^d \mathbf{C}^2$ be the space of degree d polynomials in 2 variables. We know that such a polynomial always factors as a product of d linear forms. Fix p such that $2 \le p \le d$, and let Y be the set of polynomials such that some linear form appears with multiplicity $\ge p$.

Pick a basis x, y for $\mathbf{C}^2$ and consider all polynomials of the form $x^p f$ where $\deg f = d - p$. This is a linear subspace, and closing this set under the action of $\mathbf{GL}_2(\mathbf{C})$ gives Y. Set $X = \mathbf{P}^1$, and let $\mathcal{S}$ be the subbundle of $U \times X$ given by pairs $([\ell], \ell^p f)$. Then the image of $\mathcal{S}$ in U is Y. Note that a general polynomial in Y will be of the form $\ell^p f$ where ℓ does not divide f, so the map is again a birational isomorphism. $\square$

4.7. **Dynkin diagrams.** I end this portion by reformulating what we've discussed in a more uniform Lie-theoretic way. The classification of simple Lie algebras over the complex numbers can be reduced to the classification of Dynkin diagrams. They come in 4 infinite families together with 5 exceptional examples (I will omit the latter):



Each Lie algebra has a simply-connected Lie group associated to them.

- For A$_n$, it is the special linear group $\mathbf{SL}_{n+1}(\mathbf{C})$ (subgroup of determinant 1 matrices in $\mathbf{GL}_{n+1}(\mathbf{C})$).
- For B$_n$, it is the odd rank spin groups $\mathbf{Spin}_{2n+1}(\mathbf{C})$. This is a double cover of the special orthogonal group $\mathbf{SO}_{2n+1}(\mathbf{C})$. However, we can use the special orthogonal group in what follows.
- For C$_n$, it is the symplectic group $\mathbf{Sp}_{2n}(\mathbf{C})$.
- For D$_n$, it is the even rank spin group $\mathbf{Spin}_{2n}(\mathbf{C})$.

Each node i corresponds to a "fundamental representation" with highest weight ω_i, and more generally, a function f from the nodes to non-negative integers corresponds to an irreducible representation with highest weight $\sum_i f(i)\omega_i$. We translate this now to the notation we've been using with integer sequences.

- For A$_n$, $\omega_i = (1^i, 0^{n+1-i})$. Note that for $\mathbf{SL}_{n+1}(\mathbf{C})$, the determinant representation is trivial (which is why there are n nodes and not $n+1$). The only difference between representations of $\mathbf{SL}_{n+1}(\mathbf{C})$ and $\mathbf{GL}_{n+1}(\mathbf{C})$ is that highest weights for the latter are weakly decreasing sequences in $\mathbf{Z}^{n+1}$ while for the former we take $\mathbf{Z}^{n+1}$ modulo the subgroup spanned by (1^{n+1}).
- For B$_n$, and $i < n$, $\omega_i = (1^i, 0^{n-i})$. In our previous notation, $\omega_n = \frac{1}{2}(1^n)$. So in particular, the irreducible representations of $\mathbf{SO}_{2n+1}(\mathbf{C})$ have highest weights $\sum_i f(i)\omega_i$ where $f(n)$ is even. The cases where $f(i)$ is odd correspond to representations of $\mathbf{Spin}_{2n+1}(\mathbf{C})$ that do not factor through $\mathbf{SO}_{2n+1}(\mathbf{C})$.
- For C$_n$, $\omega_i = (1^i, 0^{n-i})$.
- For D$_n$ and $i \le n-2$, $\omega_i = (1^i, 0^{n-i})$. In our previous notation, $\omega_{n-1} = \frac{1}{2}(1^n)$ and $\omega_n = \frac{1}{2}(1^{n-1}, -1)$. So the irreducible representations of $\mathbf{SO}_{2n}(\mathbf{C})$ have highest weights $\sum_i f(i)\omega_i$ where $f(n-1)+f(n)$ is even, and the odd case corresponds to representations of $\mathbf{Spin}_{2n}(\mathbf{C})$ that do not factor through $\mathbf{SO}_{2n+1}(\mathbf{C})$.

In all cases, n refers to the dimension of any maximal torus. There will be a standard maximal torus and each node corresponds to a pair of "root subgroups", one denoted "positive" and one "negative". In the case of $\mathbf{SL}_{n+1}(\mathbf{C})$, the positive root subgroup of the ith node is the set of matrices with 1's on the diagonal, an arbitrary scalar in position $(i, i+1)$, and 0's elsewhere. The negative one is the transpose of this.

The positive root subgroups together with the standard maximal torus generate a standard Borel subgroup (taking the negative ones instead gives an "opposite" Borel subgroup). Conjugacy classes of parabolic subgroups are indexed by sets of nodes not equal to the whole graph. Explicitly, given a set of nodes, the subgroup generated by the standard Borel subgroup and the corresponding negative root subgroups is a parabolic subgroup whose conjugacy class we take.

Hence, homogeneous spaces are classified by proper subsets of the nodes. In our examples, they always indexed certain kinds of flags, and the nodes can be identified with subspaces of a particular kind.

- For A_n, the ith node corresponds to i-dimensional subspaces.
- For B_n and C_n, the ith node corresponds to i-dimensional isotropic subspaces.
- For D_n, if $i \le n-2$, then the ith node corresponds to i-dimensional isotropic subspaces. The $(n-1)$st and nth nodes correspond to the two types of n-dimensional isotropic subspaces.

In almost all cases, the condition to be a flag is that subspaces are nested when appropriate (the exception is for D_n when dealing with nodes $n-1$ and n, in which case their intersection should be of dimension $n-1$).

For small values of n, the diagrams in these families overlap, which leads to isomorphisms of the Lie algebras, their simply-connected Lie groups, the corresponding homogeneous spaces, and the corresponding irreducible representations.

Example 4.7.1. We have $\mathrm{B}_2 = \mathrm{C}_2$. This tells us that $\mathbf{Spin}_5(\mathbf{C}) \cong \mathbf{Sp}_4(\mathbf{C})$. The first fundamental representation of $\mathbf{Spin}_5(\mathbf{C})$ is the vector representation $\mathbf{C}^5$ and that matches with the second fundamental representation of $\mathbf{Sp}_4(\mathbf{C})$ which is $\bigwedge^2 \mathbf{C}^4/\langle\Omega\rangle$. In particular, the latter representation has an orthogonal form on it which is compatible with the $\mathbf{Sp}_4(\mathbf{C})$-action.

This also implies that $\mathbf{OFl}(1;5) \cong \mathbf{IFl}(2;4)$. The former is a quadric hypersurface in $\mathbf{P}^4$. By semisimplicity, $\bigwedge^2 \mathbf{C}^4/\langle\Omega\rangle$ can be realized as a subspace in $\bigwedge^2 \mathbf{C}^4$ and being in the ordinary Grassmannian means that the Pfaffian vanishes, which is quadratic. So this isomorphism tells us that being in the complement of $\langle\Omega\rangle$ and having 0 Pfaffian suffices to belong to $\mathbf{IFl}(2;4)$. $\square$

Example 4.7.2. We have $\mathrm{A}_3 = \mathrm{D}_3$. In that case, the first fundamental representation of D_3, which is an orthogonal space $\mathbf{C}^6$ matches up with the second fundamental representation of A_3, which is $\bigwedge^2 \mathbf{C}^4$. The orthogonal form on the latter is given by exterior multiplication: given $\alpha, \beta \in \bigwedge^2 \mathbf{C}^4$, define $\omega(\alpha,\beta)$ to be the coefficient of $e_1 \wedge e_2 \wedge e_3 \wedge e_4$ in $\alpha \wedge \beta$. The condition $\alpha \wedge \alpha = 0$ translates to $\operatorname{rank} \alpha \le 2$, so we see that the quadric hypersurface in $\mathbf{P}^5$ matches up with the Grassmannian $\mathbf{Gr}(2,4)$.

Since node 1 of A_3 goes to node 2 of D_3, this says that $\mathbf{P}^3 \cong \mathbf{OFl}(3;6)$, and similarly $\mathbf{P}^3 \cong \mathbf{Gr}(3,4) \cong \mathbf{OFl}(3';6)$. If we mark both outer nodes, we get an isomorphism $\mathbf{Fl}(1,3;4) \cong \mathbf{OFl}(3,3';6)$, where the latter can equivalently be described as the space of 2-dimensional isotropic subspaces. $\square$

Example 4.7.3. Note that $\mathrm{A}_n = \mathrm{A}_n$ in a non-trivial way: we can reverse the order of the nodes, so that node i matches with node $n+1-i$. This amounts to matching up $\bigwedge^i \mathbf{C}^{n+1}$ with $(\bigwedge^{n+1-i} \mathbf{C}^{n+1})^*$. These two are isomorphic as $\mathbf{SL}_{n+1}(\mathbf{C})$-representations (though differ by a power of the determinant representation as $\mathbf{GL}_{n+1}(\mathbf{C})$-representations). This also realizes the isomorphism $\mathbf{Gr}(i, n+1) \cong \mathbf{Gr}(n+1-i, n+1)$.

Similarly, D_n has a non-trivial automorphism that swaps $n-1$ and n. From what we've said before, if n is odd, this amounts to swapping a representation for its dual. ☐

5. Classical invariant theory

5.1. **Schur–Weyl duality.** Let V be a finite-dimensional vector space. The tensor power $V^{\otimes n}$ is a representation of $\mathbf{GL}(V)$, and it is also a representation of the symmetric group $\mathfrak{S}_n$ by permuting tensors. More specifically, to get a left action, we have

$$\sigma \cdot (v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}.$$

Define $\mathrm{End}(W) = \mathrm{Hom}(W, W)$. Consider the space of linear operators $U = \mathrm{End}(V^{\otimes n})$. Then U is a $\mathbf{C}$-algebra where addition is the usual addition, and multiplication is composition of linear maps. The action of $\mathbf{GL}(V)$ gives a function $\mathbf{GL}(V) \to U$. Let A be the linear subspace spanned by the image. This is a subalgebra of U. Similarly, let B be the linear span of the image of $\mathfrak{S}_n \to U$, which is also a subalgebra.

Given a subalgebra $R \subset \mathrm{End}(W)$ for any vector space W, we define its **commutant** $R^! = \{x \in U \mid xr = rx \text{ for all } r \in R\}$.

For our purposes, a finite-dimensional $\mathbf{C}$-algebra is **semisimple** if it is isomorphic to a direct product of matrix algebras. This is equivalent to asking that every finite-dimensional representation is semisimple.

Theorem 5.1.1 (Double commutant theorem)**.** *Let $R \subset \mathrm{End}(W)$ be a semisimple subalgebra.*

(1) $S = R^!$ is semisimple and $R = S^!$.
(2) As an $R \times S$-representation, $W = \bigoplus_{i \in I} M_i \otimes N_i$ where M_i are irreducible representations of R, and the N_i are irreducible representations of S. Furthermore, $M_i \not\cong M_j$ and $N_i \not\cong N_j$ for $i \neq j$.

Proof. Since R is semisimple, we have a decomposition $W = \bigoplus_{i \in I} M_i^{\oplus m_i}$ where M_i are the different irreducible representations of R and m_i are their multiplicities. Then we can identify $R = \prod_{i \in I} \mathrm{End}(M_i)$. By Schur's lemma, $S = R^!$ is identified with a product of matrix algebras $\prod_{i \in I} \mathrm{End}(\mathbf{C}^{m_i})$ and in particular is semisimple. The ith component acts on $M_i^{\oplus m_i} \cong (\mathbf{C}^{m_i})^{\oplus \dim M_i}$ where $\mathbf{C}^{m_i}$ is the standard representation on column vectors and acts by 0 on all other components. Applying Schur's lemma again shows that $S^! = \prod_{i \in I} \mathrm{End}(\mathbf{C}^{\dim M_i})$. But $R \subset S^!$ and they have the same dimension, so we get equality.

The second part follows since the algebra of $n \times n$ has a unique irreducible representation given by column vectors $\mathbf{C}^n$. The product structure of R and S then gives the non-isomorphism statements. ☐

We have $B \subseteq A^!$ and $A \subseteq B^!$ since the actions commute with each other.

Proposition 5.1.2. *$B^! = A$.*

Proof. We have an identification

$$U = \mathrm{End}(V^{\otimes n}) = (\mathrm{End}\, V)^{\otimes n}$$

given as follows: if $A_1 \otimes \cdots \otimes A_n$ is a simple tensor on the right side, then we send it to the linear map $V^{\otimes n} \to V^{\otimes n}$ given again on simple tensors by $v_1 \otimes \cdots \otimes v_n \mapsto A_1(v_1) \otimes \cdots \otimes A_n(v_n)$. Under this identification, the image of $g \in \mathbf{GL}(V)$ in $(\mathrm{End}\, V)^{\otimes n}$ is $g \otimes \cdots \otimes g$.

Using this description, we claim that $A = \varphi \in (\operatorname{End} V)^{\otimes n}$ such that $\sigma\varphi = \varphi$ for all $\sigma \in \mathfrak{S}_n$, i.e., the linear space of symmetric tensors. Let f be a linear function on the space of symmetric tensors. Then we get a degree n polynomial F on $\operatorname{End}(V)$ via $F(A) = f(A \otimes \cdots \otimes A)$ and this is an isomorphism between linear functions and degree n polynomials. Since $\mathbf{GL}(V) \subset \operatorname{End}(V)$ is Zariski dense, F is identically 0 if and only if $F(g) = 0$ for all $g \in \mathbf{GL}(V)$. In particular, if f is any linear function on the space of symmetric tensors which is 0 on the subspace A, then $f = 0$, which implies that A is the entire space.

Finally, we translate what it means to be symmetric, i.e., commute with all permutations to $\operatorname{End}(V^{\otimes n})$. Pick $A_1 \otimes \cdots \otimes A_n \in (\operatorname{End} V)^{\otimes n}$ and $\sigma \in \mathfrak{S}_n$. Then $\sigma(A_1 \otimes \cdots \otimes A_n) = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)}$ corresponds to the linear map $v_1 \otimes \cdots \otimes v_n \mapsto A_{\sigma^{-1}(1)}(v_1) \otimes \cdots \otimes A_{\sigma^{-1}(n)}(v_n)$. We can rewrite this last expression as

$$\sigma \cdot (A_1(v_{\sigma(1)}) \otimes \cdots \otimes A_n(v_{\sigma(n)})) = \sigma(A_1 \otimes \cdots \otimes A_n)\sigma^{-1}(v_1 \otimes \cdots \otimes v_n).$$

So being symmetric in $(\operatorname{End} V)^{\otimes n}$ corresponds to commuting with all permutations in $\operatorname{End}(V^{\otimes n})$, which shows that $A = B^!$. □

Corollary 5.1.3. *$A^! = B$.*

Proof. Recall that representations of finite groups are semisimple over the complex numbers, so this means that B, which is a quotient of the group algebra of $\mathfrak{S}_n$, is also semisimple. So we can apply the double commutant theorem. □

Corollary 5.1.4. *As $\mathfrak{S}_n \times \mathbf{GL}(V)$-representations, we have*

$$V^{\otimes n} \cong \bigoplus_\lambda \mathbf{S}^\lambda \otimes \mathbf{S}_\lambda(V)$$

where λ ranges over all partitions of size n and $\ell(\lambda) \leq \dim V$, and the $\mathbf{S}^\lambda$ are irreducible representations of $\mathfrak{S}_n$.

The $\mathbf{S}^\lambda$ are called **Specht modules**.

Proof. This follows from the double commutant theorem. It remains to find the indexing set I in that theorem. We know from Pieri's rule that $V^{\otimes n}$ is a sum of Schur functors $\mathbf{S}_\lambda(\mathbf{C}^n)$ where λ is as in the theorem. □

Another consequence of Pieri's rule is that $\dim \mathbf{S}^\lambda$ is the set of sequences of adding boxes to the empty partition to get λ such that each intermediate shape is a Young diagram. If we put the number i in the ith box, we get a **standard Young tableau** (a special kind of semistandard Young tableau). Hence, $\dim \mathbf{S}^\lambda$ is the number of standard Young tableaux of shape λ.

Next, we would like to describe the analogue of this result when $\mathbf{GL}(V)$ is replaced by either $\mathbf{O}(V)$ or $\mathbf{Sp}(V)$. This is known as the first fundamental theorem of invariant theory (FFT) for these groups.

5.2. **FFT for classical groups.** Here we follow [P, Chapter 11].

We start with a slightly different setup. Let E and V be vector spaces with $\dim V = m$ and $\dim E = n$, and let $G \subset \mathbf{GL}(V)$ be a subgroup. We will consider the ring of invariants

$$\operatorname{Sym}(V \otimes E)^G = \bigoplus_\lambda \mathbf{S}_\lambda(V)^G \otimes \mathbf{S}_\lambda(E)$$

where the equality comes from the Cauchy identity. Since $\mathbf{GL}(E)$ commutes with $\mathbf{GL}(V)$ (and hence G), we see that the invariant ring has an action of $\mathbf{GL}(E)$.

Lemma 5.2.1. *If $\dim V = m$ and $n \geq m$, then $\mathrm{Sym}(V \otimes \mathbf{C}^n)^G$ is generated by $\mathrm{Sym}(V \otimes \mathbf{C}^m)^G$ together with the action of $\mathbf{GL}_n\mathbf{C}$.*

Proof. The sum is over all partitions λ such that $\ell(\lambda) \leq \min(\dim V, \dim E)$. In particular, if $\dim E > \dim V$, let $E' \subset E$ be a subspace with $\dim E' = \dim V$. Then $\mathbf{S}_\lambda E' \subseteq \mathbf{S}_\lambda E$ and the latter is generated by $\mathbf{S}_\lambda E'$ under $\mathbf{GL}(E)$ (since we can arrange a choice of Borel subgroup so that the subspace contains a highest weight vector of $\mathbf{GL}(E)$). □

If $G \subseteq \mathbf{SL}(V)$, i.e., $\det g = 1$ for all $g \in G$, then the determinants of $m \times m$ submatrices are all invariants.

Lemma 5.2.2. *If $G \subseteq \mathbf{SL}(V)$ and $n \geq m$, then $\mathrm{Sym}(V \otimes \mathbf{C}^n)^G$ is generated by $\mathrm{Sym}(V \otimes \mathbf{C}^{m-1})^G$ together with the $m \times m$ determinants, and the action of $\mathbf{GL}_n\mathbf{C}$.*

Proof. When $\lambda = (1^m)$, we have $\mathbf{S}_\lambda V = \bigwedge^m V$ is the determinant representation. Furthermore, for any λ with $\ell(\lambda) = m$ we can write $\lambda = (k^m) + \mu$ where $\ell(\mu) < m$, and the highest weight vector of $\mathbf{S}_\lambda V \otimes \mathbf{S}_\lambda \mathbf{C}^m$ is a product of the kth power of highest weight vector of $\bigwedge^m V \otimes \bigwedge^m \mathbf{C}^m$ and the highest weight vector of $\mathbf{S}_\mu V \otimes \mathbf{S}_\mu \mathbf{C}^m$. Now use the previous result. □

Now consider the orthogonal group $G = \mathbf{O}(V)$ with orthogonal form ω. We can use this to make the identification $V \cong V^*$: a vector $v \in V$ becomes the linear functional $u \mapsto \omega(u, v)$. This gives an invariant subspace of $\mathrm{Sym}^2 V$ and hence there is an invariant subspace in $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 E$. Under our identification, this is spanned by the invariant functions $\varphi_{i,j}$ for $1 \leq i \leq j \leq n$ given by

$$\varphi_{i,j}(\sum_{k=1}^n v_k \otimes e_k) = \omega(v_i, v_j).$$

If $j < i$, we set $\varphi_{i,j} = \varphi_{j,i}$.

Theorem 5.2.3 (FFT for $\mathbf{SO}, \mathbf{O}$). *Let $\dim V = m$.*

1. $\mathrm{Sym}(V \otimes \mathbf{C}^n)^{\mathbf{SO}(V)}$ *is generated as a $\mathbf{C}$-algebra by the $\varphi_{i,j}$ and, if $n \geq m$, the $m \times m$ determinants.*
2. $\mathrm{Sym}(V \otimes \mathbf{C}^n)^{\mathbf{O}(V)}$ *is generated as a $\mathbf{C}$-algebra by the $\varphi_{i,j}$.*

Proof. Let R be the ring generated by the $\varphi_{i,j}$ and the $m \times m$ determinants. Note that R is closed under the action of $\mathbf{GL}_n(\mathbf{C})$.

Next, we show that (1) implies (2). From the above it suffices to assume $m = n$. Since $\mathbf{O}(V)$ acts on the $m \times m$ determinants by ± 1, (1) implies that the ring of $\mathbf{O}(V)$ invariants is the subring of R generated by the $\varphi_{i,j}$ and products of an even number of the $m \times m$ determinants. If u is the generic $m \times m$ matrix whose entries are $V \otimes \mathbf{C}^m$, then

$$u^T I' u = (\varphi_{i,j})_{i,j=1,\dots,n} = \Phi.$$

In particular, $\det \Phi = (-1)^m \det(u)^2$. Since $\det \Phi$ is generated by the $\varphi_{i,j}$, this shows that products of an even number of the $m \times m$ determinant are redundant as generators for the $\mathbf{O}(V)$-invariant ring and hence (2) follows from (1).

Now we prove that both (1) and (2) hold by induction on m. By the above discussion, it suffices to prove (1) but we will assume that both hold for $m - 1$. If $m = 1$, then $\mathbf{SO}(V)$ is the trivial group and $\mathrm{Sym}(V \otimes \mathbf{C}^m) \cong \mathbf{C}[t]$ where t is the 1×1 determinant.

Now assume $m > 1$. Since $\mathbf{SO}(V) \subset \mathbf{SL}(V)$, by the previous lemma we may assume that $n = m-1$. Pick a vector $u \in V$ of norm 1, i.e., $\omega(u,u) = 1$ and let V' be its orthogonal complement. Then V' is also an orthogonal space which we think of as both a quotient and subspace of V. Furthermore, any $g' \in \mathbf{O}(V')$ can be extended to an element $g \in \mathbf{SO}(V)$ by defining $g(u) = \det(g')u$. Hence, if $f \in \mathrm{Sym}(V \otimes \mathbf{C}^{m-1})^{\mathbf{SO}(V)}$, then its restriction $f' \in \mathrm{Sym}(V' \otimes \mathbf{C}^{m-1})$ is an $\mathbf{O}(V)$-invariant. By induction, this means that f' is a polynomial in the $\varphi_{i,j}$. Let $F \in \mathrm{Sym}(V \otimes \mathbf{C}^{m-1})$ be the same polynomial in the $\varphi_{i,j}$.

We claim that $F = f$. If u' is any other norm 1 vector, then there exists $g \in \mathbf{SO}(V)$ such that $g(u) = u'$ and hence the restriction of $F - f$ to $\mathrm{Sym}(V'' \otimes \mathbf{C}^{m-1})$, where V'' is the orthogonal complement of u', is 0. Next, if we identify $V \otimes \mathbf{C}^{m-1} = V^{\oplus m-1}$, then the set of $(v_1, \dots, v_{m-1})$ such that their span is a non-degenerate $(m-1)$-dimensional subspace is Zariski dense (being degenerate means that either they don't have $(m-1)$-dimensional span, or else that some vector v_i has zero pairing with all others, and both conditions are Zariski closed). By what we have just argued, $F = f$ on this Zariski dense set, so they are equal. □

Now consider the tensor power $V^{\otimes n}$. If $n = 2k$ is even, pick a perfect matching M of $\{1, \dots, n\}$, i.e., write $\{1, \dots, n\}$ as a disjoint union of 2 element subsets. Then we get a $\mathbf{O}(V)$-invariant linear function $V^{\otimes n} \to \mathbf{C}$ by

$$\varphi_M(v_1 \otimes \cdots \otimes v_n) = \prod_{\{i,j\} \in M} \omega(v_i, v_j).$$

We can identity this functional with an element of $V^{\otimes n}$ using our identification $V = V^*$ from before.

Corollary 5.2.4. *$(V^{\otimes n})^{\mathbf{O}(V)}$ is 0 if n is odd and otherwise is spanned by φ_M over all choices of perfect matchings M.*

Proof. We have $\mathrm{Sym}(V \otimes \mathbf{C}^n) = \mathrm{Sym}(V^{\oplus n}) = \mathrm{Sym}(V)^{\otimes n}$ which contains $V^{\otimes n}$ as a linear subspace. Under the action of the maximal torus $T \subset \mathbf{GL}_n\mathbf{C}$, $V^{\otimes n}$ is the (1^n)-weight space. From FFT, the $\mathbf{O}(V)$-invariants of $\mathrm{Sym}(V \otimes \mathbf{C}^n)$ is generated by $\varphi_{i,j}$, which is a weight vector for T of weight $(0, \dots, 1, \dots, 1, \dots 0)$ with 1's in positions i and j. Hence the invariants in $V^{\otimes n}$ is spanned by all products of $\varphi_{i,j}$ whose weight adds up to (1^n), which is only possible if n is even, and in that case, we get the φ_M. □

Now consider the case $G = \mathbf{Sp}(V)$ with symplectic form ω. Again we can identify $V = V^*$ using this form and we get an invariant subspace in $\bigwedge^2 V \otimes \bigwedge^2 E$ which is spanned by invariant functions $\psi_{i,j}$ for $1 \le i < j \le n$ given by

$$\psi_{i,j}(\sum_{k=1}^n v_k \otimes e_k) = \omega(v_i, v_j).$$

If $j < i$, we set $\psi_{i,j} = -\psi_{j,i}$.

Theorem 5.2.5 (FFT for **Sp**)**.** *$\mathrm{Sym}(V \otimes \mathbf{C}^n)^{\mathbf{Sp}(V)}$ is generated as a $\mathbf{C}$-algebra by the $\psi_{i,j}$.*

Proof. Let $\dim V = 2m$.

Let R be the ring generated by the $\psi_{i,j}$. Then R is closed under the action of $\mathbf{GL}_n(\mathbf{C})$ and hence it suffices to consider the case $n = 2m$. In that case, let u be the generic matrix. We have

$$u^T \Omega u = (\psi_{i,j}) = \Psi$$

where Ψ is skew-symmetric. Then $\mathrm{Pf}(\Psi) = \det u \cdot \mathrm{Pf}\,\Omega = \pm \det u$. Since $\mathrm{Pf}(\Psi) \in R$, we see that R also contains the $2m \times 2m$ determinant functions, and so we may further assume that $n = 2m - 1$.

We proceed by induction on m. If $m = 1$, then our ring is $\mathrm{Sym}(V)$, and each $\mathrm{Sym}^k(V)$ for $k > 0$ is an irreducible representation of $\mathbf{Sp}(V)$, so R is just the constants, so there is nothing to prove.

Let $v_1, \dots, v_{2m}$ be a symplectic basis for V, let V' be the span of $v_2, \dots, v_{2m}$ and let W be the span of $v_2, \dots, v_{2m-1}$. We will think of V' and W as both subspaces and quotients of V. Then W is a symplectic space of dimension $2m - 2$. Let $f \in \mathrm{Sym}(V \otimes \mathbf{C}^{2m-1})^{\mathbf{Sp}(V)}$. We claim that the restriction of f to $\mathrm{Sym}(V' \otimes \mathbf{C}^{2m-1})$ does not depend on the coordinates involving v_{2m}. To see this, for $t \in \mathbf{C} \setminus 0$, consider $g_t \in \mathbf{Sp}(V)$ which is the identity on W and $g(v_1) = t^{-1}v_1$, $g(v_{2m}) = tv_{2m}$. Then the restriction of f is invariant under g_t for all t, but since it does not use v_1, it will get scaled by powers of t if it involves any v_{2m}, and hence it must not involve v_{2m}.

Now restrict f to $\mathrm{Sym}(W \otimes \mathbf{C}^{2m-1})$. Since $\mathbf{Sp}(W) \subset \mathbf{Sp}(V)$, by induction we have that f is a polynomial in the $\psi_{i,j}$. Let $F \in \mathrm{Sym}(V \otimes \mathbf{C}^{2m-1})$ be the same polynomial. By what we argued, the restriction of $F - f$ to $\mathrm{Sym}(V' \otimes \mathbf{C}^{2m-1})$ is 0. The rest follows by a Zariski density argument as before. $\square$

Now consider the tensor power $V^{\otimes n}$. If $n = 2k$ is even, pick a perfect matching M of $\{1, \dots, n\}$, i.e., write $\{1, \dots, n\}$ as a disjoint union of 2 element subsets. Then we get a $\mathbf{Sp}(V)$-invariant linear function $V^{\otimes n} \to \mathbf{C}$ by

$$\psi_M(v_1 \otimes \cdots \otimes v_n) = \prod_{\{i,j\} \in M} \omega(v_i, v_j).$$

We can identity this functional with an element of $V^{\otimes n}$ using our identification $V = V^*$ from before.

Corollary 5.2.6. *$(V^{\otimes n})^{\mathbf{Sp}(V)}$ is 0 if n is odd and otherwise is spanned by ψ_M over all choices of perfect matchings M.*

5.3. **Brauer algebras.** Let G be either the orthogonal group or symplectic group for V and consider $V^{\otimes n}$. Define $\varepsilon = 1$ for the orthogonal case and $\varepsilon = -1$ for the symplectic case. Let A be the linear subspace of $\mathrm{End}(V^{\otimes n})$ spanned by the elements of G. We want to compute $B = A^!$. We can reduce this to FFT as follows. We have

$$\mathrm{End}(V^{\otimes n}) = (V^{\otimes n})^* \otimes V^{\otimes n} = V^{\otimes 2n}$$

where the form ω gives the identification $V^* = V$. Then $A^!$ corresponds to the linear G-invariants of $V^{\otimes 2n}$, which is spanned by functions coming from perfect matchings. Let M be a perfect matching of $\{1, \dots, 2n\}$. Under this identification, it makes sense to distinguish $\{1, \dots, n\}$ from $\{n+1, \dots, 2n\}$.

Given a perfect matching M, the corresponding endomorphism β_M can be defined "locally" on an element $v_1 \otimes \cdots \otimes v_n$ as follows.

- For all edges $\{i, j\}$ with $i, j \le n$, we remove v_i and v_j and scale the result by $\omega(v_i, v_j)$.
- Next, if there is an edge between $\{i, j+n\}$ with $i, j \le n$, then we move v_i to the jth tensor position. The set of all edges forms a permutation between some of the bottom vertices and some of the top vertices. In the symplectic case, we also scale by the sign of this permutation.

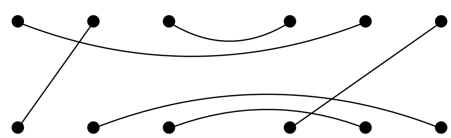
- Finally, for all edges $\{i+n, j+n\}$ with $i, j \le n$, we insert the element Ω in positions i and j, where $\Omega \in V^{\otimes 2}$ is the sum over dual basis vectors. To be precise, for $\mathbf{O}(m)$ with hyperbolic basis $\{e_1, \dots, e_m\}$, $\Omega = \sum_{i=1}^m e_i \otimes e_{m+1-i}$. For $\mathbf{Sp}(2m)$ with symplectic basis $\{e_1, \dots, e_{2m}\}$, $\Omega = \sum_{i=1}^m (-e_i \otimes e_{2m+1-i} + e_{2m+1-i} \otimes e_i)$.

Theorem 5.3.1. *The centralizer $B = A^!$ is linearly spanned by β_M as M ranges over all perfect matchings of $\{1, \dots, 2n\}$.*

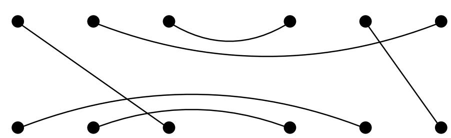
What is the product structure on the β_M? In general, they are not linearly independent, so it will be easier to define an algebra abstractly for which the β_M are a basis and to describe its multiplication.

We first define a monoid $\mathcal{B}_n$ on the set of perfect matchings of $\{1, \dots, 2n\}$. We think of $\{1, \dots, 2n\}$ as 2 rows of n ordered dots and the subset $\{i, j\}$ is represented by an edge between these two dots. The product MM' of matchings M and M' is obtained by stacking on M' on top of M, identifying the 2 middle rows, and picking out the resulting edges between dots not in the middle row. We also set $n(M, M')$ to be the number of connected components of the graph involving only the middle row.

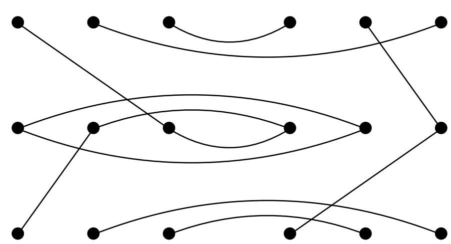
To illustrate, we give an example of composition in $\mathcal{B}_6$. Let M be the matching



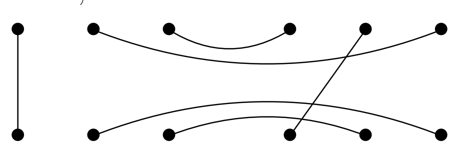
and let M' be the matching



After putting M above M', we obtain the graph



There is one component that only touches middle vertices, and so $n(M, M') = 1$. Discarding it and ignoring the middle vertices, we are left with



and this is MM'.

Finally, given $\delta \in \mathbf{C}$, the Brauer algebra $\mathbf{B}_n(\delta)$ is the $\mathbf{C}$-algebra with basis given by the perfect matchings of $\{1, \dots, 2n\}$ with multiplication given by

$$M \cdot M' = \delta^{n(M,M')} MM'$$

where the product on the right is the product in $\mathcal{B}_n$. (If $\delta = n(M, M') = 0$, we interpret $0^0 = 1$.)

Proposition 5.3.2. *We have a surjective homomorphisms*

$$\mathbf{B}_n(\dim V) \to \mathrm{End}_{\mathbf{O}(V)}(V^{\otimes n}), \qquad \mathbf{B}_n(-\dim V) \to \mathrm{End}_{\mathbf{Sp}(V)}(V^{\otimes n})$$

which in both cases sends the basis vector M to β_M.

As is, it's not so easy to show that multiplication is preserved. It's easier to start with a presentation of the Brauer algebra by generators and relations and show directly that all of the relations behave correctly, but we will omit this, since understanding the multiplication won't be crucial for the following discussion.

Remark 5.3.3. The strange sign conventions in the symplectic case are needed. For example, $(1,2)\Omega = -\Omega$, but in the Brauer algebra, permuting the ends of a "cap" leaves it invariant. So to compensate, permutations act by an additional sign. □

Remark 5.3.4. The image of the Brauer algebra is $A^!$ which is semisimple by the double commutant theorem. For Schur–Weyl duality, the group algebra of the symmetric group was already semisimple. However, in general, $\mathbf{B}_n(\delta)$ can fail to be semisimple when $\delta \in \mathbf{Z}$. Nonetheless, since it factors through a semisimple quotient, the action of $\mathbf{B}_n(\pm\dim V)$ on $V^{\otimes n}$ is semisimple. □

Here is a variation. For each $i < j$, we have a map $\omega_{i,j} \colon V^{\otimes n} \to V^{\otimes(n-2)}$ such that $\omega_{i,j}(v_1 \otimes \cdots \otimes v_n)$ is $\omega(v_i, v_j)$ times the result of removing v_i and v_j. We let $V^{[n]} = \bigcap_{i<j} \ker \omega_{i,j}$ and call it the space of **traceless tensors**.

Theorem 5.3.5. *The commutant of the image of G in $\mathrm{End}(V^{[n]})$ is the linear span of the symmetric group and hence as a $\mathfrak{S}_n \times G$-representation, we have*

$$V^{[n]} \cong \bigoplus_\lambda \mathbf{S}^\lambda \otimes \mathbf{S}_{[\lambda]}V$$

where the sum is over all partitions λ of size n with $2\ell(\lambda) \le \min(\dim V, n)$.

Proof. By restricting to the traceless tensors, the perfect matchings which have edges $\{i, j\}$ with $i, j \le n$ now act by 0. The remaining perfect matchings can be identified with permutations of n, and hence we see that the linear span of $\mathfrak{S}_n$ is the commutant of the linear span of G.

As for the indexing set, the restrictions are necessary, so it suffices to show that these terms actually appear. By Schur–Weyl duality, it suffices to show that there is a traceless tensor in $\mathbf{S}_\lambda V$, but since $2\ell(\lambda) \le \dim V$, we know its highest weight vector with respect to the Borel subgroup for the standard hyperbolic basis uses only $e_1, \dots, e_d$ where $d \le \dim V/2$, and there are no nonzero pairings between these vectors. □

Corollary 5.3.6. *As a $\mathbf{B}_n(\pm\dim V) \times G$-representation, we have*

$$V^{\otimes n} \cong \bigoplus_\lambda W^\lambda \otimes \mathbf{S}_{[\lambda]}(V)$$

where λ ranges over all partitions such that $2\ell(\lambda) \le \dim V$, $n \ge |\lambda|$, and $n - |\lambda|$ is even, and the W^λ are distinct irreducible $\mathbf{B}_n(\pm\dim V)$-representations.

Proof. We already know that the pair of algebras are commutants of one another, so it remains to understand the indexing set. We have a surjective map $\omega_{1,2} \colon V^{\otimes n} \to V^{\otimes(n-2)}$ whose kernel contains $V^{[n]}$, and hence by induction, we see that $\mathbf{S}_{[\lambda]}V$ appears in the decomposition if λ satisfies the conditions stated. □

6. Spin groups

Now we turn to the question of constructing the spinor representations, which we have alluded to a few times as being the missing representations in the theory of the orthogonal groups. The standard construction uses Clifford algebras, which we explain first.

6.1. Clifford algebras.

Let V be a finite-dimensional orthogonal space with form ω. First, the **tensor algebra** $T(V) = \bigoplus_{d\geq 0} V^{\otimes d}$ is an associative algebra with product given by concatenation of tensors, i.e.,

$$(v_1 \otimes \cdots \otimes v_d) \cdot (w_1 \otimes \cdots \otimes w_e) = v_1 \otimes \cdots \otimes v_d \otimes w_1 \otimes \cdots \otimes w_e.$$

The **Clifford algebra** $C(V,\omega) = C(V)$ is the quotient of $T(V)$ by the 2-sided ideal generated by the relations

$$vw + wv = 2\omega(v,w).$$

Here $v, w \in V^{\otimes 1}$ and the right hand side is in $\mathbf{C} = V^{\otimes 0}$. So the relations are not homogeneous with respect to the natural $\mathbf{Z}$-grading, but they are homogeneous with respect to the natural $\mathbf{Z}/2$-grading, so we have a decomposition

$$C(V) = C^+(V) \oplus C^-(V)$$

where $C^+(V)$ is the image of $\bigoplus_{d\geq 0} V^{\otimes 2d}$ and $C^-(V)$ is the image of $\bigoplus_{d\geq 0} V^{\otimes(2d+1)}$. The elements of $C^+(V)$ are **even** and the elements of $C^-(V)$ are **odd** and every element can uniquely be written as a sum of an even and odd element. Furthermore, multiplication of homogeneous elements has the expected behavior (even times even is even, etc.)

We will make use of the following *universal property* of $C(V)$: If A is any associative $\mathbf{C}$-algebra, then to specify a homomorphism $\varphi\colon C(V) \to A$, it is enough to specify $\varphi(v)$ for all $v \in V$ subject to the fact that they satisfy $\varphi(v)\varphi(w) + \varphi(w)\varphi(v) = 2\omega(v,w)$.

Lemma 6.1.1. *Let $v_1, \dots, v_m$ be a basis for V. For $I = (i_1, \dots, i_k)$ with $1 \leq i_1 < \cdots < \cdots i_k \leq m$, define $v_I = v_{i_1} \cdots v_{i_k}$ (with $v_\varnothing = 1$). Then the v_I form a basis for $C(V)$ and hence* $\dim C(V) = 2^{\dim V}$.

Proof. The products of the v_i span $T(V)$ and hence span $C(V)$. The relations tell us that if we see two vectors $v_i v_j$ out of order (i.e., $i > j$), then we can replace it by $2\omega(v_i, v_j) - v_j v_i$, so that we can rewrite any such product as a linear combination of products where all of the vectors are in order. Hence the v_I span $C(V)$.

Next, for each d, define $C^{\leq d}(V)$ to be the span of all elements $w_1 \cdots w_k$ where $w_i \in V$ and $k \leq d$. Then we have an isomorphism of vector spaces

$$C(V) \cong \bigoplus_{d\geq 0} C^{\leq d}(V)/C^{\leq d-1}(V).$$

The elements in $C^{\leq d}(V)/C^{\leq d-1}(V)$ satisfy the relations for $\bigwedge^d V$ (since our basic relation becomes just $vw + wv = 0$) which we know has a basis given by v_I with $|I| = d$. Hence they are linearly independent and the same is true in $C(V)$. □

The Clifford algebras turn out to be semisimple algebras, but the decomposition into matrix algebras depends on the parity of m. We first consider the case when $m = 2n$ is even. In that case, write $V = W \oplus W'$ where W, W' are n-dimensional isotropic subspaces. For instance, with respect to a hyperbolic basis $e_1, \dots, e_{2n}$, W is the span of $e_1, \dots, e_n$ and W' is the span of $e_{n+1}, \dots, e_{2n}$. Note that W' is canonically isomorphic to W^* via ω.

We define $\bigwedge^\bullet(W) = \bigoplus_{d\geq 0} \bigwedge^d W$. This is an associative algebra (the **exterior algebra**) via concatenation of tensors.

Theorem 6.1.2. *If $\dim V$ is even, then we have an isomorphism $C(V) \cong \mathrm{End}(\bigwedge^\bullet W)$, i.e., $C(V)$ is isomorphic to the algebra of matrices of size 2^n. In particular, $C(V)$ is a simple algebra.*

Proof. We first construct a homomorphism $\varphi\colon C(V) \to \mathrm{End}(\bigwedge^\bullet W)$ using the universal property of $C(V)$.

For each $w \in W$, let $\varphi(w)$ denote left multiplication by w in the exterior algebra, i.e., $\varphi(w)(\alpha) = w \wedge \alpha$. For $w' \in W'$, we define

$$\varphi(w')(w_1 \wedge \cdots \wedge w_d) = \sum_{i=1}^{d} (-1)^{i-1} 2\omega(w', w_i)(w_1 \wedge \cdots \widehat{w_i} \cdots \wedge w_d)$$

where $\widehat{w_i}$ means that we remove that factor. For a general $v \in V$, we write $v = w + w'$ with $w \in W$ and $w' \in W'$ and define $\varphi(v) = \varphi(w) + \varphi(w')$.

We have to check that they satisfy $\varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 2\omega(v, v')$. By linearity, it is enough to check this when v, v' are elements of either W or W'.

- If $v, v' \in W$, then $\varphi(v)$ and $\varphi(v')$ anticommute and $\omega(v, v') = 0$.
- Similarly, if $v, v' \in W'$, a direct calculation also shows that $\varphi(v)$ and $\varphi(v')$ anticommute and $\omega(v, v') = 0$.
- Finally, assume $v \in W$ and $v' \in W'$. Then

$$\varphi(v)\varphi(v')(w_1 \wedge \cdots \wedge w_d) = \sum_{i=1}^{d} (-1)^{i-1} 2\omega(v', w_i)(v \wedge w_1 \wedge \cdots \widehat{w_i} \cdots \wedge w_d)$$

$$\varphi(v')\varphi(v)(w_1 \wedge \cdots \wedge w_d) = 2\omega(v', v) w_1 \wedge \cdots \wedge w_d + \sum_{i=1}^{d} (-1)^{i} 2\omega(v', w_i)(v \wedge w_1 \wedge \cdots \widehat{w_i} \cdots \wedge w_d).$$

When we add them, the two sums cancel, so $\varphi(v)\varphi(v') + \varphi(v')\varphi(v) = 2\omega(v, v')$.

Next, we show that φ is injective. Suppose that $a = \sum_I \alpha_I e_I \in \ker\varphi$ where I ranges over all subsets of $\{1, \dots, 2n\}$. We will show that all $\alpha_I = 0$ by induction on $s(I) = \#(I \cap \{n+1, \dots, 2n\})$. For the base case, we have $\varphi(a)(1) = \sum_I \alpha_I e_I$ where the sum is over all I such that $s(I) = 0$. Since these are a basis for $\bigwedge^\bullet W$, we see that $\alpha_I = 0$ for all such I.

In general, assuming we know it is true for $s(I) \leq d$, we consider $\varphi(a)(e_{i_1} \wedge \cdots \wedge e_{i_{d+1}})$ over all choices $i_1 < \cdots < i_{d+1}$. Then $\varphi(e_I)$ applied to this element is 0 if $s(I) > d+1$ or if $s(I) = d+1$ and $I \neq \{i_1, \dots, i_{d+1}\}$. If $I = \{i_1, \dots, i_{d+1}\}$, then we $\varphi(e_I)(e_{i_1} \wedge \cdots \wedge e_{i_{d+1}})$ is a nonzero constant c_I, and so we get

$$\varphi(a)(e_{i_1} \wedge \cdots \wedge e_{i_{d+1}}) = c_{i_1, \dots, i_{d+1}} \sum_{I \subseteq \{1, \dots, n\}} \alpha_{I \cup \{i_1, \dots, i_{d+1}\}} e_I.$$

Again, the e_I are linearly independent in $\bigwedge^\bullet W$, so $\alpha_{I \cup \{i_1, \dots, i_{d+1}\}} = 0$ for all I.

Since φ is injective, and $\dim C(V) = 2^{2n} = (2^n)^2 = \dim \mathrm{End}(\bigwedge^\bullet W)$, we conclude that φ is an isomorphism. $\square$

Next, define $\bigwedge^{\mathrm{even}}(W) = \bigoplus_{d\geq 0} \bigwedge^{2d} W$ and $\bigwedge^{\mathrm{odd}}(W) = \bigoplus_{d\geq 0} \bigwedge^{2d+1} W$.

Corollary 6.1.3. $C^+(V) \cong \mathrm{End}(\bigwedge^{\mathrm{even}} W) \times \mathrm{End}(\bigwedge^{\mathrm{odd}} W)$ *so $C^+(V)$ is also semisimple.*

Proof. The action of $C^+(V)$ on $\bigwedge^\bullet W$ preserves both of these subspaces, so we get a map

$$\varphi\colon C^+(V) \to \mathrm{End}(\bigwedge^{\text{even}} W) \times \mathrm{End}(\bigwedge^{\text{odd}}(W)$$

which is injective since it is the restriction of the injective map φ on $C(V)$. Since

$$\dim C^+(V) = \dim C(V)/2 = 2^{2n-1} = 2 \cdot (2^{n-1})^2 = \dim(\mathrm{End}(\bigwedge^{\text{even}} W) \times \mathrm{End}(\bigwedge^{\text{odd}}(W)),$$

this map is also an isomorphism. □

Now we consider the odd case $m = 2n+1$. In that case, we write $V = W \oplus W' \oplus L$ where W, W' are n-dimensional isotropic subspace as before, and $L = (W \oplus W')^\perp$. With respect to a hyperbolic basis $e_1, \dots, e_{2n+1}$, W is the span of $e_1, \dots, e_n$, L is the span of e_{n+1}, and W' is the span of $e_{n+2}, \dots, e_{2n+1}$.

Theorem 6.1.4. *If* $\dim V$ *is odd, then we have an isomorphism* $C(V) \cong \mathrm{End}(\bigwedge^\bullet W) \times \mathrm{End}(\bigwedge^\bullet W')$*, i.e.,* $C(V)$ *is isomorphic to the product of two algebras of matrices of size* 2^n *each. In particular,* $C(V)$ *is a semisimple algebra.*

Furthermore, $C^+(V) \cong \mathrm{End}(\bigwedge^\bullet W) \cong \mathrm{End}(\bigwedge^\bullet W')$*.*

Proof. We proceed as in the even case to define $\varphi\colon C(V) \to \mathrm{End}(\bigwedge^\bullet W)$. For $v \in W \oplus W'$, we define $\varphi(v)$ in exactly the same way. Pick $\ell \in L$ so that $\omega(\ell, \ell) = 1$. Then we define $\varphi(\ell)(w_1 \wedge \cdots \wedge w_d) = (-1)^d w_1 \wedge \cdots \wedge w_d$. Then for any $w \in W \oplus W'$, we have $\varphi(\ell)\varphi(w) = -\varphi(w)\varphi(\ell)$ since $\varphi(w)$ changes the parity of every homogeneous element of $\bigwedge^\bullet W$. Also $\omega(w, \ell) = 0$ by definition. So φ is a homomorphism.

We define $\varphi'\colon C(V) \to \mathrm{End}(\bigwedge^\bullet W')$ in exactly the same way except we swap the roles of W and W' and we multiply $\varphi(\ell)$ by $(-1)^n$. Combining them gives us a homomorphism $\Phi\colon C(V) \to \mathrm{End}(\bigwedge^\bullet W) \times \mathrm{End}(\bigwedge^\bullet W')$. For a subset $I \subseteq \{1, \dots, 2n+1\} \setminus \{n+1\}$, let $I' = I \cup \{n+1\}$. Suppose $a = \sum_I (\alpha_I e_I + \alpha_{I'} e_{I'}) \in \ker \Phi$. Then $a \in \ker \varphi$ and $a \in \ker \varphi'$. Using an argument similar to the previous one, we can show that $\alpha_I + \alpha_{I'} = 0$ and $\alpha_I - \alpha_{I'} = 0$ by induction on $\#(I \cap \{n+2, \dots, 2n+1\})$ (we omit the details) and hence Φ is injective. By a dimension count, Φ is an isomorphism.

Now consider the action of $C^+(V)$ on $\bigwedge^\bullet W)$. Since every basis element of $C^+(V)$ is either a basis element of $C^+(W \oplus W')$ or ℓ times such a basis element, the argument in the even case implies the last statement. □

We will need the following statement later, whose proof we leave as an exercise.

Lemma 6.1.5.
- *If* $x \in C^+(V)$ *and* $xv = vx$ *for all* $v \in V$*, then* $x \in C^0(V)$ *is a scalar.*
- *If* $x \in C^-(V)$ *and* $xv = -vx$ *for all* $v \in V$*, then* $x = 0$*.*

6.2. **Spin groups.** The Clifford algebra $C(V)$ has a linear map $x \mapsto x^*$ defined by

$$(v_1 \cdots v_d)^* = (-1)^d v_d \cdots v_1$$

for $v_1, \dots, v_d \in V$. It is an anti-involution: $(x^*)^* = x$ and $(xy)^* = y^* x^*$. To see this is well-defined, define $C(V)^{\mathrm{op}}$ to be the opposite algebra of $C(V)$, i.e., it has a multiplication $x \cdot y = yx$ where the right side is multiplication in $C(V)$. Then we need to check that $*$ satisfies the universal property. Pick $v, w \in V$. Then

$$v^* \cdot w^* + w^* \cdot v^* = (-w)(-v) + (-v)(-w) = 2\omega(v, w).$$

We also define an algebra homomorphism $\alpha\colon C(V) \to C(V)$ by

$$\alpha(v_1 \cdots v_d) = (-1)^d v_1 \cdots v_d.$$

We define

$$\mathbf{Pin}(V) = \{x \in C(V) \mid xx^* = 1,\ xvx^* \in V \text{ for all } v \in V\},$$
$$\mathbf{Spin}(V) = \{x \in C^+(V) \mid xx^* = 1,\ xvx^* \in V \text{ for all } v \in V\}.$$

Then both are clearly groups under multiplication, which we call the **pin group** and **spin group**, respectively. Thinking of them as a subsets of matrix algebra(s), the conditions are given by the polynomial equations, and hence both are algebraic groups. Note that we have a group homomorphism $\mathbf{Pin}(V) \to \mathbf{Z}/2$ which takes the degree of an element and the kernel is $\mathbf{Spin}(V)$. We will see in the proof below that $\mathbf{Pin}(V)$ contains odd elements, so $\mathbf{Spin}(V)$ is an index 2 subgroup of $\mathbf{Pin}(V)$.

Given $x \in \mathbf{Pin}(V)$, we define $\rho_x \in \mathbf{GL}(V)$ by $\rho_x(v) = \alpha(x)vx^*$. The latter is an element of $V \subset C(V)$. Then ρ is a homomorphism.

Theorem 6.2.1. *The image of* $\mathbf{Pin}(V)$ *under* ρ *is* $\mathbf{O}(V)$ *and* $\ker \rho = \{\pm 1\}$. *The image of* $\mathbf{Spin}(V)$ *under* ρ *is* $\mathbf{SO}(V)$.

Proof. First we need to show that $\rho_x \in \mathbf{O}(V)$, i.e., $\omega(\rho_x(v), \rho_x(w)) = \omega(v, w)$ for all $v, w \in V$. Note that $v = -v^*$ for any $v \in V$. Now use the Clifford relation:

$$\begin{aligned}
2\omega(\rho_x(v), \rho_x(w)) &= \rho_x(v)\rho_x(w) + \rho_x(w)\rho_x(v) \\
&= -\rho_x(v)(\rho_x(w))^* - \rho_x(w)(\rho_x(v))^* \\
&= -\alpha(x)vx^*xw^*\alpha(x)^* - \alpha(x)wx^*xv^*\alpha(x)^* \\
&= -\alpha(x)vw^*\alpha(x)^* - \alpha(x)wv^*\alpha(x)^* \\
&= \alpha(x)(-vw^* - wv^*)\alpha(x)^* \\
&= 2\omega(v, w)\alpha(xx^*) = 2\omega(v, w)
\end{aligned}$$

Next, we need to show that ρ is surjective onto $\mathbf{O}(V)$. By a previous homework, it was shown that $\mathbf{O}(V)$ is generated by reflections s_a for $a \in V$ such that $\omega(a, a) \neq 0$ where

$$s_a(v) = v - \frac{2\omega(v, a)}{\omega(a, a)} a$$

Note that $s_a = s_{\lambda a}$ for any nonzero $\lambda \in \mathbf{C}$, so we may assume that $\omega(a, a) = -1$ in the above generating set. Given that, in the Clifford algebra we have $aa^* = -a^2 = -\omega(a, a) = 1$ and $axa^* = (2\omega(a, x) - xa)a^* = -2\omega(a, x)a - x \in V$, so that $a \in \mathbf{Pin}(V)$. Furthermore,

$$\rho_a(v) = \alpha(a)va^* = -ava^* = 2\omega(a, v)a + v = s_a(v)$$

and hence ρ is surjective.

Finally, we compute $\ker \rho$. Pick $x \in \ker \rho$ and write $x = x_0 + x_1$ as a sum of even and odd elements. Since ρ_x is the identity, for all $v \in V$, we have $v = \alpha(x)vx^*$, or equivalently, $vx = \alpha(x)v$. Expanding this, we get $vx_0 + vx_1 = x_0v - x_1v$, and matching homogeneous components, we get $vx_0 = x_0v$ and $vx_1 = -x_1v$. By Lemma 6.1.5, x_0 is a scalar and $x_1 = 0$. But then $xx^* = x_0^2 = 1$ so $x_0 = \pm 1$ and hence $\ker \rho = \{\pm 1\}$.

This shows that if $g \in \mathbf{O}(V)$ and $g = s_{a_1} \cdots s_{a_r}$, then $\rho^{-1}(g) = \{\pm a_1 \cdots a_r\}$. Since $\mathbf{SO}(V)$ consists of products of an even number of reflections, we see that $\rho^{-1}(\mathbf{SO}(V)) \subseteq C^+(V)$ and hence $\rho^{-1}(\mathbf{SO}(V)) \subseteq \mathbf{Spin}(V)$. Since both are index 2 subgroups of $\mathbf{Pin}(V)$, we get equality and hence $\rho(\mathbf{Spin}(V)) = \mathbf{SO}(V)$. $\square$

Remark 6.2.2. One can show that $\mathbf{Spin}(V)$ is a connected group. Since the Euclidean topology refines the Zariski topology, it suffices to show that it is connected in the Euclidean topology. Since $\rho(\mathbf{Spin}(V)) = \mathbf{SO}(V)$ is connected, it suffices to show that any two elements in a fiber are connected by a path, and via translation, it suffices to do 1 and -1. For that, pick $v, w \in V$ such that $\omega(v, v) = \omega(w, w) = -1$ and $\omega(v, w) = 0$ and define for $t \in [0, \pi/2]$

$$t \mapsto (\cos(t)v + \sin(t)w)(\cos(t)v - \sin(t)w)$$

Then this is a path from 1 to -1 in $\mathbf{Spin}(V)$ (we omit the check that it satisfies the conditions). $\square$

Proposition 6.2.3. *$\bigwedge^\bullet W$ is an irreducible representation of $\mathbf{Pin}(V)$.*

If $\dim V$ is odd, then it remains irreducible for $\mathbf{Spin}(V)$.

If $\dim V$ is even, then $\bigwedge^{\mathrm{even}} W$ and $\bigwedge^{\mathrm{odd}} W$ are irreducible representations for $\mathbf{Spin}(V)$ which are not isomorphic.

We call $\bigwedge^\bullet W$ the **spinor representation**, and $\bigwedge^{\mathrm{even}} W$ and $\bigwedge^{\mathrm{odd}} W$ are the **half-spinor representations**.

Proof. First, note that $\mathbf{Pin}(V)$ linearly spans $C(V)$: from the previous proof, if we pick a basis $x_1, \dots, x_m$ of V so that $\omega(x_i, x_i) = -1$, then $x_I \in C(V)$ and they form a basis of $C(V)$. Then $\bigwedge^\bullet W$ is an irreducible representation of $C(V)$ and hence the same is true for $\mathbf{Pin}(V)$. The statements about $\mathbf{Spin}(V)$ amount to the behavior of $C^+(V)$ in each of the cases of m even and m odd. $\square$

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