Math 251 C, Lecture 9

Note Title 4/17/2020

Thm (Cauchy identity) in $\Lambda(n,m)[[t]]$:

$$\prod_{i=1}^{n}\prod_{j=1}^{m}(1-x_iy_jt)^{-1} = \sum_{\lambda} s_\lambda(x_1,\dots,x_n)\, s_\lambda(y_1,\dots,y_m)\, t^{|\lambda|}$$

sum over all $\lambda=(\lambda_1\geq\lambda_2\geq\dots\geq\lambda_r\geq 0)$

$r=\min(n,m)$

Pf. From before

$$\mathrm{Sym}(\mathbb{C}^n\otimes\mathbb{C}^m) \cong \bigoplus_\lambda S_\lambda\mathbb{C}^n\otimes S_\lambda\mathbb{C}^m$$

(as $GL_n\mathbb{C}\times GL_m\mathbb{C}$ reps)

char $\Big\downarrow$

$$\sum_{d\geq 0} h_d(\underbrace{x_1y_1, x_2y_1,\dots,x_ny_m}_{\text{all choices of } x_iy_j})\,t^d = \sum_\lambda s_\lambda(x_1..x_n)\,s_\lambda(x_1..x_m)\,t^{|\lambda|}$$

$\square$

Rmk. The isom from before is equivalent to this symmetric polynomial identity.

Direct proof of identity is possible (RSK correspondence)

Thm $$\prod_{1\le i\le j\le n}(1-x_ix_jt)^{-1} = \sum_\lambda s_{2\lambda}(x_1,\ldots,x_n)\,t^{|\lambda|}$$

$$\prod_{1\le i<j\le n}(1-x_ix_jt)^{-1} = \sum_\lambda s_{(2\lambda)^T}(x_1,\ldots,x_n)\,t^{|\lambda|}.$$

Pf. First one comes from $\mathrm{Sym}(\mathrm{Sym}^2\mathbb{C}^n)$

second one comes from $\mathrm{Sym}(\Lambda^2\mathbb{C}^n)$ $\square$

Solution to particular plethysm:

$$S_{(d)}(S_{(2)}\mathbb{C}^n) = \mathrm{Sym}^d(\mathrm{Sym}^2\mathbb{C}^n) \cong \bigoplus_{|\lambda|=d} S_{2\lambda}\mathbb{C}^n$$

$$S_{(d)}(S_{(1,1)}\mathbb{C}^n) = \mathrm{Sym}^d(\Lambda^2\mathbb{C}^n) \cong \bigoplus_{|\lambda|=d} S_{(2\lambda)^T}\mathbb{C}^n.$$

Symmetric Functions

$\pi_n : \Lambda(n+1) \longrightarrow \Lambda(n)$ (ring homomorphism)

$f(x_1, \ldots x_{n+1}) \longrightarrow f(x_1, \ldots x_n, 0)$

(can compare reps of $GL_{n+1}\mathbb{C}$ & $GL_n \mathbb{C}$).

$$\pi_n s_\lambda(x_1, \ldots x_{n+1}) = s_\lambda(x_1, \ldots x_n)$$

Λ = graded inverse limit of $\{\pi_n : \Lambda(n+1) \to \Lambda(n)\}$

Explicitly, a degree d element $f \in \Lambda$ is a sequence (f_n) $f_n \in \Lambda(n)$ is homogeneous of deg d s.t. $\pi_n(f_{n+1}) = f_n$.

A general element of Λ is a finite sum of degree d elements.

Alternatively, an element of Λ is a bounded degree power series in $x_1, x_2, \ldots$ which is symmetric in all variables.

<u>Ex.</u> $f_n = x_1 + x_2 + \cdots + x_n$ $\quad \pi_n(f_{n+1}) = f_n$

$\underline{f = x_1 + x_2 + x_3 + \ldots}$

$f_n = s_\lambda(x_1, \ldots x_n)$ limit is called s_λ

Schur function

Λ is a ring, has basis $\{s_\lambda\}$

Have specialization maps $\Lambda \to \Lambda(n)$

$$f \to f(x_1, \ldots x_n, 0, 0, \ldots)$$

Existence of Λ tells us that we can do computations in Λ and get answers for $\Lambda(n)$ $\forall n$

Heuristic: polynomial representations of $GL_n \mathbb{C}$ exhibit stability wrt n

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<u>Multiplying Schur polynomials.</u>

Given μ, ν partitions, we have expressions

$$s_\mu(x_1 \ldots x_n) s_\nu(x_1, \ldots, x_n) = \sum_\lambda \underbrace{c^\lambda_{\mu\nu}(n)}_{\in \mathbb{Z}} s_\lambda(x_1 \ldots x_n)$$

<u>Note:</u> $$S_\mu \mathbb{C}^n \otimes S_\nu \mathbb{C}^n \cong \bigoplus_\lambda (S_\lambda \mathbb{C}^n)^{\oplus c^\lambda_{\mu\nu}(n)}$$

$$\Rightarrow c^\lambda_{\mu\nu}(n) \geq 0$$

Can also do same thing w/ sym. fcns:

$$s_\mu s_\nu = \sum_\lambda \underbrace{c^\lambda_{\mu\nu}}_{\in \mathbb{Z}} s_\lambda$$

Apply specialization $\Lambda \to \Lambda(n)$:

$$s_\mu(x_1,\dots x_n)\, s_\nu(x_1,\dots x_n) = \sum_\lambda c^\lambda_{\mu\nu}\, s_\lambda(x_1,\dots x_n)$$

$$\Longrightarrow c^\lambda_{\mu\nu} = c^\lambda_{\mu\nu}(n) \quad \forall n$$

<u>Def</u>. $c^\lambda_{\mu\nu}$ <u>Littlewood-Richardson coefficients</u>

<u>Properties</u>:
- $c^\lambda_{\mu\nu} \neq 0 \Longrightarrow |\lambda| = |\mu| + |\nu|$
  $\Longrightarrow \mu \subseteq \lambda$ & $\nu \subseteq \lambda$
- $c^\lambda_{\mu\nu} = c^\lambda_{\nu\mu}$
- $c^\lambda_{\mu\nu} = c^{\lambda^T}_{\mu^T\nu^T}$

Special cases: $\nu = (k)$, $\nu = (1^k)$

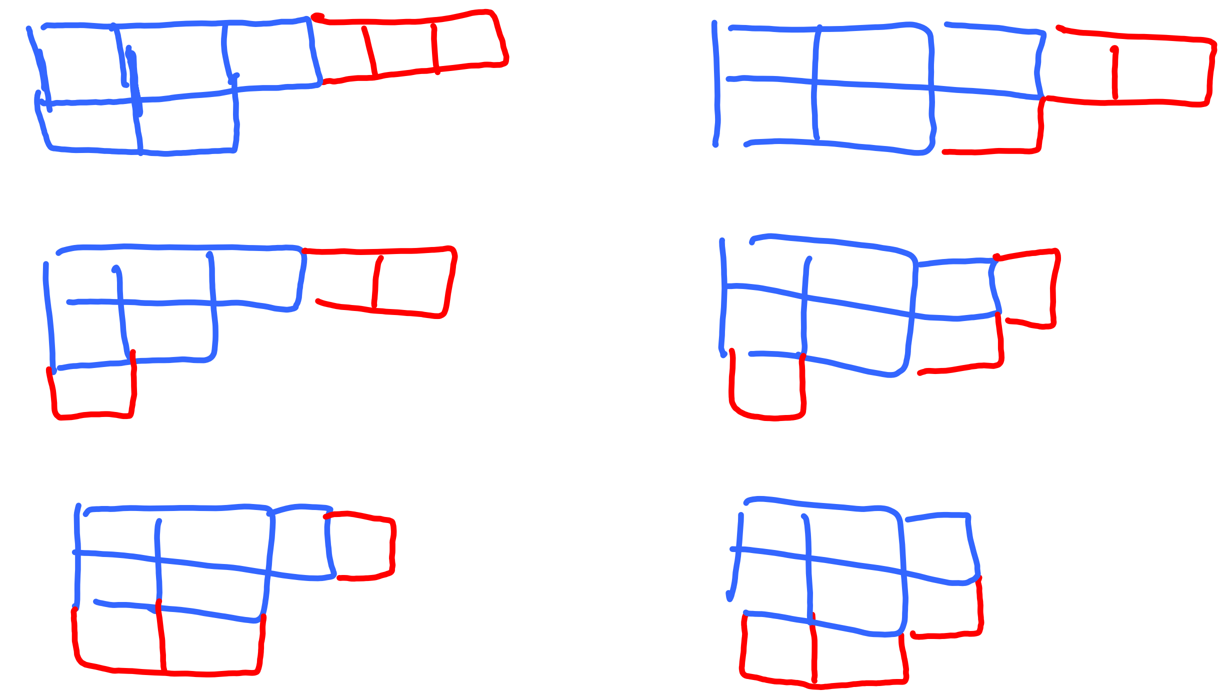
tensoring w/ Sym^k tensoring w/ Λ^k

<u>Thm</u> (Piéri rule)

$$s_\mu s_k = \sum_{\substack{\lambda \text{ s.t.}\\ |\lambda| = |\mu| + k \\ \lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots}} s_\lambda, \qquad s_\mu s_{(1^k)} = \sum_{\substack{\lambda \text{ s.t.}\\ |\lambda| = |\mu| + k \\ \lambda_1^T \geq \mu_1^T \geq \lambda_2^T \geq \mu_2^T \geq \dots}} s_\lambda$$

Ex. $\mu = (3,2)$, $k = 3$

Find all λ s.t. $|\lambda| = 8$, $\lambda_1 \geq 3 \geq \lambda_2 \geq 2 \geq \lambda_3 \geq 0$



$$s_{32} s_3 = s_{62} + s_{53} + s_{521} + s_{431} + s_{422} + s_{332}$$

Given $\mu \subseteq \lambda$, call skew shape λ/μ a horizontal strip (HS) if no column contains more than 1 box.

Similarly, λ/μ is a vertical strip (VS), if no row contains more than 1 box.

Thm (Pieri rule)

$$s_\mu s_k = \sum_{\substack{\lambda \text{ s.t.} \\ \lambda/\mu \text{ is HS} \\ \text{of size } k}} s_\lambda \qquad s_\mu s_{(1^k)} = \sum_{\substack{\lambda \text{ s.t.} \\ \lambda/\mu \text{ is VS} \\ \text{of size } k}} s_\lambda$$

Symplectic Groups

Recall: $GL_m\mathbb{C} \curvearrowright \Lambda^2\mathbb{C}^m$ ← space of skew-symmetric $m\times m$ matrices

via $g\cdot u = (g^{-1})^T u g^{-1}$

Assume $m=2n$ even.

__Def.__ Given a full rank matrix $u\in\Lambda^2\mathbb{C}^m$, the corresponding symplectic group is its stabilizer, i.e.,

$$\{g\in GL_m\mathbb{C} \mid (g^{-1})^T u g^{-1} = u\}$$
$$= \{g\in GL_m\mathbb{C} \mid g^T u g = u\}$$

Since any 2 full rank matrices are in the same $GL_m\mathbb{C}$-orbit, any 2 symplectic groups are conjugate in $GL_m\mathbb{C}$, in particular isomorphic.

$$I'_n = \underbrace{\begin{pmatrix} 0 & & 1 \\ & ⋰ & \\ 1 & & 0 \end{pmatrix}}_{n} \qquad \Omega_n = \begin{pmatrix} 0 & I'_n \\ -I'_n & 0 \end{pmatrix} \in \Lambda^2\mathbb{C}^{2n}$$

omit indices usually

$$Sp_{2n}\mathbb{C} = \mathrm{stab}(\Omega)$$

$$= \{ g \mid g^T \Omega g = \Omega \}$$

$$= \{ g \mid \Omega^{-1} (g^T)^{-1} \Omega = g \} \quad (\text{since } \Omega^{-1} = -\Omega^T)$$