Math 251 C, Lecture 7

Note Title 4/13/2020

U = space of $n\times n$ skew-symmetric matrices
$= \{u \mid u^T = -u\}$

Rep of $GL_n\mathbb{C}$ via: $g\cdot u = (g^{-1})^T u g^{-1}$.

Let f_i be the Pfaffian of the upper left $2i \times 2i$ submatrix of u for $i = 1, \dots, \lfloor \frac{n}{2} \rfloor$

- $Pf(X)^2 = \det(X)$ for X skew-symmetric
- $Pf(gXg^T) = (\det g)\, Pf(X)$

<u>Prop</u>. f_i is a highest weight vector of weight $(\underbrace{1, 1, \dots, 1}_{2i}, 0, \dots 0)$.

Pf omitted (similar to previous case)

Define $J_n \in U$ as follows:

if n even: $J_n = \begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & & \\ & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} & & \\ & & \ddots & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix}} \end{pmatrix}$

if n odd: $J_n = \begin{pmatrix} J_{n-1} & 0 \\ 0 & 0 \end{pmatrix}$

Prop. The B-orbit of J_n is open and dense.

Pf omitted

Prop. If λ is weight of a highest weight vector in $\mathbb{C}[U]$, then λ_i^T is even for all i and $\lambda_n \geq 0$ (alternatively, $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \ldots$)

Pf is similar to previous cases.

Thm. We have an isomorphism of $GL_n\mathbb{C}$-reps:

$$\mathbb{C}[U] = \mathrm{Sym}(\textstyle\bigwedge^2 \mathbb{C}^n) \cong \bigoplus_{\lambda} S_{(2\lambda)^T}(\mathbb{C}^n).$$

all partitions $\frac{n}{2} \geq \lambda_1 \geq \lambda_2 \geq \cdots$

Pf. Take products of $f_1, \ldots, f_{\lfloor n/2 \rfloor}$ to get all h.w. vectors. $\square$

Schur functors

Let $\lambda = (\lambda_1, \ldots, \lambda_r)$ be partition

$\mu = \lambda^T = (\mu_1, \ldots, \mu_s)$ transpose partition

V = vector space

Two maps: ① $\bigwedge^d V \longrightarrow V^{\otimes d}$

$$v_1 \wedge \cdots \wedge v_d \longmapsto \sum_{\sigma \in \mathfrak{S}_d} (\operatorname{sgn} \sigma)\, v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(d)}$$

(extend linearly to all elements of $\bigwedge^d V$)

② $V^{\otimes d} \longrightarrow \mathrm{Sym}^d V$ (extend linearly)

$$v_1 \otimes \cdots \otimes v_d \longmapsto v_1 \cdots v_d$$

The Schur functor $S_\lambda(V)$ is the image of:

$$\bigwedge^{\mu_1} V \otimes \bigwedge^{\mu_2} V \otimes \cdots \otimes \bigwedge^{\mu_s} V \xrightarrow{①} V^{\otimes \mu_1} \otimes V^{\otimes \mu_2} \otimes \cdots \otimes V^{\otimes \mu_s}$$

$$\xrightarrow{③} V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$$

$$\xrightarrow{②} \mathrm{Sym}^{\lambda_1} V \otimes \cdots \otimes \mathrm{Sym}^{\lambda_r} V$$

③ Think of pure tensors in $V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s}$ as filling $Y(\lambda)$ w/ elements in V where $v_1 \otimes \cdots \otimes v_{\mu_i}$ in ith part go in ith column in order

$$\begin{array}{|c|}\hline v_1 \\ \hline v_2 \\ \hline \vdots \\ \hline v_{\mu_i} \\ \hline \end{array}$$

Then we reinterpret as element in $V^{\otimes \lambda_1} \otimes \cdots \otimes V^{\otimes \lambda_r}$ by pulling elements in each row.

Ex. $\lambda = (3,2)$, $\mu = (2,2,1)$ $\quad Y(\lambda) = \begin{array}{|c|c|c|}\hline & & \\ \hline & \\ \cline{1-2}\end{array}$

$\Lambda^2 V \otimes \Lambda^2 V \otimes V$

$$(v_1 \wedge v_2) \otimes (v_3 \wedge v_4) \otimes v_5 \xmapsto{①} \left(\begin{array}{|c|}\hline v_1 \\ \hline v_2 \\ \hline \end{array} - \begin{array}{|c|}\hline v_2 \\ \hline v_1 \\ \hline \end{array} \right) \otimes \left(\begin{array}{|c|}\hline v_3 \\ \hline v_4 \\ \hline \end{array} - \begin{array}{|c|}\hline v_4 \\ \hline v_3 \\ \hline \end{array} \right) \otimes v_5$$

$$= \begin{array}{|c|c|c|}\hline v_1 & v_3 & v_5 \\ \hline v_2 & v_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline v_2 & v_3 & v_5 \\ \hline v_1 & v_4 \\ \cline{1-2}\end{array} - \begin{array}{|c|c|c|}\hline v_1 & v_4 & v_5 \\ \hline v_2 & v_3 \\ \cline{1-2}\end{array} + \begin{array}{|c|c|c|}\hline v_2 & v_4 & v_5 \\ \hline v_1 & v_3 \\ \cline{1-2}\end{array}$$

$(v_1 \otimes v_3 \otimes v_5) \otimes (v_2 \otimes v_4) \cdots$

$$\xmapsto{②} v_1 v_3 v_5 \otimes v_2 v_4 - v_2 v_3 v_5 \otimes v_1 v_4 - v_1 v_4 v_5 \otimes v_2 v_3 + v_2 v_4 v_5 \otimes v_1 v_3$$

$\in \mathrm{Sym}^3 V \otimes \mathrm{Sym}^2 V$

$$S_{32} V = \mathrm{span} \langle \quad \mid v_1, v_2, v_3, v_4, v_5 \in V \rangle.$$

Facts. Maps ①, ②, ③ are compatible w/ linear maps $f: V \to W$, i.e.,

i.e.,
$$\begin{tikzcd} \bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V \arrow[r, "①"] \arrow[d] \arrow[dr, phantom, "\text{commutes}"] & V^{\otimes \mu_1} \otimes \cdots \otimes V^{\otimes \mu_s} \arrow[d] \\ \bigwedge^{\mu_1} W \otimes \cdots \otimes \bigwedge^{\mu_s} W \arrow[r, "①"] & W^{\otimes \mu_1} \otimes \cdots \otimes W^{\otimes \mu_s} \end{tikzcd}$$

vertical map:

$$v_1 \wedge \cdots \wedge v_d \mapsto f(v_1) \wedge \cdots \wedge f(v_d)$$

$$v_1 \otimes \cdots \otimes v_d \mapsto f(v_1) \otimes \cdots \otimes f(v_d)$$

In particular, ①, ②, ③ are homomorphisms of $GL(V)$-representations $\Rightarrow$ $S_\lambda V$, as we just defined it, is a $GL(V)$-representation.

<u>Thm</u> $S_\lambda V$ is irreducible and has highest weight λ.

<u>Ex</u>. $\cdot\ \lambda = (d)$, $\mu = (1, \ldots, 1)$. ① = identity on $V^{\otimes d}$

② $V^{\otimes d} \to \mathrm{Sym}^d V$ is surjective, so $S_{(d)} V = \mathrm{Sym}^d V$

$\cdot\ \lambda = (1^d)$, $\mu = (d)$ ② = identity on $V^{\otimes d}$

① is injective, so $S_{(1^d)} V = \bigwedge^d V$.

Fix basis $e_1, \dots, e_n$ for V.

A <u>tableau</u> T of shape $Y(\lambda)$ is filling of boxes of $Y(\lambda)$ w/ numbers $1, \dots, n$.

$$\tilde{e}_T := (e_{T_{1,1}} \wedge e_{T_{2,1}} \wedge \cdots \wedge e_{T_{\mu_1,1}}) \otimes \cdots \otimes (e_{T_{1,s}} \wedge \cdots \wedge e_{T_{\mu_s,s}})$$

$$\in \bigwedge^{\mu_1} V \otimes \cdots \otimes \bigwedge^{\mu_s} V.$$

e_T = image of $\tilde{e}_T$ in $S_\lambda V$.

<u>Ex.</u> $\lambda = (3,2)$. $T = \begin{array}{|c|c|c|} \hline 5 & 6 & 1 \\ \hline 4 & 4 \\ \cline{1-2} \end{array}$ $\quad \tilde{e}_T = (e_5 \wedge e_4) \otimes (e_6 \wedge e_4) \otimes e_1$

T is <u>semistandard</u> if:
- $T_{ij} \leq T_{i,j+1}$ $\forall i,j$.
- $T_{ij} < T_{i+1,j}$

i.e., weakly increase left to right in rows
strictly increase top to bottom in columns

<u>Thm</u> $\{e_T \mid T \text{ semistandard}\}$ is a basis for $S_\lambda V$.

e_T is a weight vector of weight $\mu(T)$ where

$\mu(T)_i$ = # times i appears in T

$$\Rightarrow \operatorname{char} S_\lambda V = \sum_{T \text{ semistandard}} x^{\mu(T)}$$

Schur polynomial
$s_\lambda(x_1, \dots, x_n)$

Highest weight vector comes from tableau

$$T_0: \begin{array}{llll} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2 & \cdots & 2 \\ 3 & \cdots & 3 \\ \vdots \end{array}$$

put $T_{ij} = i$

weight $= \lambda$.

Can show directly that $\tilde{e}_{T_0}$ is h.w. vector and homomorphisms preserve highest weight vectors.