Math 251C, Lecture 6

Note Title 4/10/2020

Last time: $U = n\times m$ matrices, rep of $GL_n \times GL_m$: ($m\geq n$)

$(g,h)\cdot u = (g^{-1})^T u h^{-1}$

f_i = det of upper left $i\times i$ submatrix $\in \mathbb{C}[U]$

h.w. vector of weight $(1^i, 0^{n-i}), (1^i, 0^{m-i})$

$\mathbb{C}[U]$ multiplicity-free, any h.w. (λ, λ')

satisfies $\lambda_1 = \lambda_1', \dots, \lambda_n = \lambda_n', \lambda_{n+1}' = \dots = \lambda_m' = 0, \lambda_n \geq 0$

Cor (Cauchy identity) We have an isomorphism of $GL_n\mathbb{C} \times GL_m\mathbb{C}$-representations:

$$\mathbb{C}[U] \cong \mathrm{Sym}(\mathbb{C}^n \otimes \mathbb{C}^m) \cong \bigoplus_{\lambda} S_\lambda \mathbb{C}^n \otimes S_\lambda \mathbb{C}^m$$

$\lambda_1 \geq \dots \geq \lambda_n \geq 0$

Pf ① Know $\mathbb{C}[U]$ is mult. free and every rep appearing must be of the form $S_\lambda \mathbb{C}^n \otimes S_\lambda \mathbb{C}^m$.

② Know these appear when $\lambda = (1^i)$ b/c f_i is a h.w. vector of that weight.

③ Can multiply the f_i to get new h.w. vectors

$$f_1^{d_1} f_2^{d_2} \cdots f_n^{d_n} \rightsquigarrow \text{h.w. } (d_1 + d_2 + \dots + d_n, d_2 + d_3 + \dots + d_n, \dots, d_{n-1} + d_n, d_n)$$

all partitions can be written in this form:

take $d_i = \lambda_i - \lambda_{i+1} \geq 0$ $\square$

Example 2: symmetric matrices

U = space of $n \times n$ symmetric matrices.

Rep of $GL_n\mathbb{C}$ via:

$$g \cdot u = (g^{-1})^T u g^{-1}$$

f_i = det of upper left $i \times i$ submatrix of $u \in \mathbb{C}[U]$

Prop. f_i is a h.w. vector of weight $(\underbrace{2,\dots,2}_{i},0,\dots 0)$

Pf. $g \in B$, write $g = \begin{pmatrix} g_1 & g_2 \\ 0 & g_3 \end{pmatrix}$, where g_1 is $i \times i$.

$U \ni u = \begin{pmatrix} a & b \\ b^T & c \end{pmatrix}$ where a is $i \times i$.

$$(g \cdot f_i)(u) = f_i(g^{-1} \cdot u) = f_i(g^T u g) = \det(g_1^T a g_1)$$
$$= (\det g_1)^2 f_i(u)$$

$\Rightarrow$ f_i is h.w. of weight $(2,\dots,2,0,\dots 0)$:

if $g = \begin{pmatrix} x_1 & & \\ & \ddots & \\ & & x_n \end{pmatrix}$, then $(\det g_1)^2 = x_1^2 x_2^2 \cdots x_i^2$. $\square$

Note. $f_1 \in U^*$ is h.w. of weight $(2,0,\dots,0)$

$Sym^2 \mathbb{C}^n$ is h.w. rep. of weight $(2,0,\dots,0)$

$\Rightarrow Sym^2 \mathbb{C}^n \subseteq U^*$, actually isomorphic since both have dim $\binom{n+1}{2}$.

$\Rightarrow U^* \cong Sym^2 \mathbb{C}^n$

Prop. The B-orbit Y of identity matrix I is open, dense.

Pf. $Y = \{ g^T g \mid g \text{ is upper triangular invertible} \}$

Claim: $Y = \{ u \in U \mid f_i(u) \neq 0 \text{ for } i=1,\dots,n \}$

Prove $\supseteq$ by induction on n.

$n=1$: clear

Otherwise: let u satisfy $f_i(u) \neq 0$ for $i=1..n$.

write $u = \begin{pmatrix} u' & y \\ y^T & z \end{pmatrix}$ where u' is $(n-1)\times(n-1)$

Then $f_i(u) = f_i(u')$ for $i=1,\dots,n-1$, so by induction,

$u' = h^T h$, h upper-tri, invertible $\Rightarrow$

$$\begin{pmatrix} u' & y \\ y^T & z \end{pmatrix} = \underbrace{\begin{pmatrix} h^T & 0 \\ y^T h^{-1} & \alpha \end{pmatrix}}_{= g^T} \underbrace{\begin{pmatrix} h & (h^T)^{-1} y \\ 0 & \alpha \end{pmatrix}}_{= g} \qquad \alpha^2 = z - y^T h^{-1} (h^T)^{-1} y$$

Since $f_n(u) \neq 0$, u is invertible, so

$\det(u) = \det(g)^2 \Rightarrow g$ invertible. proves claim

$\Rightarrow Y$ is open (hence dense) $\square$

Prop. If λ is a weight of a h.w. vector in $\mathbb{C}[U]$, then λ_i is even for all i, and $\lambda_n \geq 0$.

Pf. Let $H = \operatorname{stab}(I)$. We know if λ appears, then $\lambda(h) = 1 \ \forall h \in H \cap B$.

Take
$$h = \begin{array}{c} \begin{pmatrix} 1 & & & & 0 \\ & \ddots & & & \\ & & -1 & & \\ & & & \ddots & \\ 0 & & & & 1 \end{pmatrix} \\ \text{position } i \end{array}$$
Then $h^T h = I$, so $h \in H \cap B$.

$1 = \lambda(h) = (-1)^{\lambda_i} \Rightarrow \lambda_i$ even.

Finally, all weights of $\mathbb{C}[U]$ are non-negative, so we get $\lambda_n \geq 0$ for all h.w. λ in $\mathbb{C}[U]$. $\square$

Cor. We have isom. of $GL_n\,\mathbb{C}$-reps:
$$\mathbb{C}[U] = \operatorname{Sym}(\operatorname{Sym}^2 \mathbb{C}^n) \cong \bigoplus_{\lambda} S_{2\lambda} \mathbb{C}^n$$

$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

$2\lambda = (2\lambda_1, \ldots, 2\lambda_n)$

Pf. All reps must be $S_{2\lambda}\mathbb{C}^n$. But they all appear (take products of the $f_1 \ldots f_n$)

Example 3: skew-symmetric matrices

Given a vector space V, $d \geq 0$ integer, define the <u>dth exterior power</u> $\bigwedge^d V$ to be the quotient of $V^{\otimes d}$ by subspace spanned by elements of the form $v_1 \otimes \cdots \otimes v_d$ where $\exists i \neq j$ s.t. $v_i = v_j$.

Let $v_1 \wedge \cdots \wedge v_d$ be the image of $v_1 \otimes \cdots \otimes v_d$ in $\bigwedge^d V$

<u>Note</u>: $v_1 \wedge \cdots \wedge v_d = (\operatorname{sgn} \sigma)\, v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(d)} \; \forall \sigma \in \mathfrak{S}_d$.

If $e_1, \ldots e_n$ is basis for V, then
$\{e_{i_1} \wedge \cdots \wedge e_{i_d} \mid 1 \leq i_1 < i_2 < \cdots < i_d \leq n\}$ basis for $\bigwedge^d V$.

$$\Longrightarrow \dim \bigwedge^d V = \binom{\dim V}{d}.$$

U = space of $n \times n$ skew-symmetric matrices
$= \{u \mid u^T = -u\}$

Rep of $GL_n\mathbb{C}$ via: $g \cdot u = (g^{-1})^T u g^{-1}$.

Digression: Pfaffians

If u is $n \times n$ skew-sym,

$\det(u) = \det(u^T) = \det(-u) = (-1)^n \det(u)$

$\Longrightarrow$ If n odd, $\det u = 0$.

Claim: If n even, entries of u are variables, then $\det(u)$ is not irreducible!

Ex. $n=2$: $u = \begin{pmatrix} 0 & x \\ -x & 0 \end{pmatrix}$, $\det(u) = x^2$

Set $n = 2i$

Let $\Pi = \left\{ \sigma \in \mathfrak{S}_n \;\middle|\; \begin{array}{l} \sigma(1) < \sigma(3) < \cdots < \sigma(n-1) \\ \sigma(1) < \sigma(2),\ \sigma(3) < \sigma(4), \ldots, \\ \sigma(n-1) < \sigma(n) \end{array} \right\}$

$$Pf(u) := \sum_{\sigma \in \Pi} (\operatorname{sgn} \sigma)\, u_{\sigma(1),\sigma(2)}\, u_{\sigma(3),\sigma(4)} \cdots u_{\sigma(n-1),\sigma(n)}$$

is a polynomial in entries of u

Properties:
- $(Pf\, u)^2 = \det u$
- $Pf(g u g^T) = (\det g)\, Pf\, u$.

Let f_i be the Pfaffian of the upper left $2i \times 2i$ submatrix of u for $i = 1, \ldots, \lfloor \frac{n}{2} \rfloor$