Math 251C, Lecture 4

Note Title 4/6/2020

Multiplicity-Free Actions

Ref: Howe, "Perspectives on invariant theory"

Last time: irreducible reps of $GL_n\mathbb{C}$ $\qquad S_\lambda \mathbb{C}^n$

$\updownarrow \qquad\qquad \updownarrow$

$\lambda_1 \geq \cdots \geq \lambda_n \in \mathbb{Z}^n \qquad \lambda$

W rep $\Longrightarrow$ $W \cong \bigoplus_\lambda (S_\lambda \mathbb{C}^n)^{\oplus m_\lambda}$ ← multiplicity

Def. W is multiplicity-free if $m_\lambda \leq 1$ $\forall \lambda$

Similarly, if W is rep of $GL_n\mathbb{C} \times GL_m\mathbb{C}$,

Then $W \cong \bigoplus_{\lambda,\mu} (S_\lambda \mathbb{C}^n \otimes S_\mu \mathbb{C}^m)^{\oplus m_{\lambda,\mu}}$

W multiplicity-free if $m_{\lambda,\mu} \leq 1$ $\forall \lambda, \mu$.

Note: m_λ = dim of space of h.w. vectors in W of weight λ

Given vector space U, let $\mathrm{Sym}^d U^*$

= dth symmetric power of U^*

$$= (U^*)^{\otimes d} \Big/ \left\langle u_1 \otimes \cdots \otimes u_d - u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(d)} \;\middle|\; \sigma \in \mathfrak{S}_d,\ u_1, \ldots, u_d \in U^* \right\rangle$$

symmetric group

If we pick basis $x_1 \ldots x_n$ of U^*, then $\mathrm{Sym}^d U^*$ = space of homogeneous degree d polynomials in $x_1, \ldots x_n$.

$$\mathrm{Sym}\, U^* := \bigoplus_{d \geq 0} \mathrm{Sym}^d U^*$$

= all polynomials in $x_1 \ldots x_n$.

<u>Note</u>: If $GL_k \mathbb{C}$ acts on U, then $\mathrm{Sym}^d U^*$ & also $\mathrm{Sym}\, U^*$ are representations of $GL_k \mathbb{C}$

If $f \in \mathrm{Sym}\, U^*$, $u \in U$, then evaluation $f(u) \in \mathbb{C}$ makes sense.

$\mathrm{Sym}\, U^*$ is infinite-dim'l, but each $\mathrm{Sym}^d U^*$ is finite-dim'l, so previous results apply.

<u>Ex</u> $GL(V)$ acting on $U = V^*$, so $U^* = V$

<u>Claim</u>: $\mathrm{Sym}^d V$ is irreducible for all $d \geq 0$.

Pick a basis $x_1 \ldots x_n$ for V

weight vectors of $\mathrm{Sym}^d V$ are monomials in $x_1 \ldots x_n$. $x_1^{d_1} \ldots x_n^{d_n} \longleftrightarrow$ weight $(d_1, \ldots, d_n)$

x_1^d is h.w., and only one, weight is $(d, 0, 0, 0)$

<u>Eg</u>. $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \cdot x_1 x_2 = (a x_1)(b x_1 + c x_2)$

$\Longrightarrow$ $\mathrm{Sym}\, V$ is multiplicity-free.

<u>Goal</u>: Find condition on U so that $\mathrm{Sym}\, U^*$ is multiplicity-free.

Zariski topology

U = vector space, $f: U \to \mathbb{C}$ is polynomial if it is so wrt a basis for U.

Given a set I of polynomials $f_i: U \to \mathbb{C}$

$$Z(I) := \{ u \in U \mid f(u) = 0 \ \forall f \in I \}$$

(zero set of I)

WLOG, we usually assume that I is an ideal:
1. $f+g \in I \ \forall f, g \in I$
2. $fh \in I \ \forall f \in I$, h arbitrary polynomial
3. $I \neq \emptyset$.

Given a set of polynomials $\{f_i\}$,

$\langle f_i \rangle$ = smallest ideal containing all f_i

$= \{ h_1 f_1 + \dots + h_r f_r \mid h_i \text{ arbitrary} \}$

I is generated by $\{f_i\}$ if $I = \langle f_i \rangle$.

Thm (Hilbert basis thm) Every ideal can be generated by finite set of polynomials.

Sum of ideals: $I_1 + I_2 = \{f+g \mid f \in I_1, g \in I_2\}$

Product of ideals: $I_1 I_2 = \langle fg \mid f \in I_1, g \in I_2 \rangle$

(infinite sums of ideals ok)

Def. The Zariski topology on U is the topology whose closed sets are the $Z(I)$.

Check: ① $\emptyset = Z(\langle 1 \rangle)$

② $U = Z(\langle 0 \rangle)$

③ Intersection: $\bigcap_j Z(I_j) = Z(\sum_j I_j)$

④ Finite unions: $Z(I_1) \cup \cdots \cup Z(I_r) = Z(I_1 \cdots I_r)$.

The $Z(I)$ are affine varieties.

Given affine variety $X \subset U$, denote $\mathbb{C}[X]$ the quotient of $\operatorname{Sym} U^*$ by ideal of all polynomials which are identically 0 on X.

Note: $\mathbb{C}[U] = \operatorname{Sym} U^*$

Ex. $GL_n\mathbb{C} \subset \mathbb{C}^{n^2}$ is complement of affine variety $Z(\langle \det \rangle) \Rightarrow GL_n\mathbb{C}$ is open in Zariski topology

Introduce new variable t

Consider $Z(\langle t \cdot \det - 1 \rangle) \subset \mathbb{C}^{n^2+1}$

$$\{(g, \lambda) \mid \lambda \cdot \det(g) = 1\} = \{(g, \lambda) \mid \lambda = \frac{1}{\det g}\}$$

$g \in \mathbb{C}^{n^2}$, $\lambda \in \mathbb{C}$

λ is redundant

projecting onto first n^2 coordinates, we get a bijection $Z(\langle t \cdot \det - 1 \rangle) \xrightarrow{\sim} \mathbb{C}^{n^2} \setminus Z(\langle \det \rangle)$

$\Rightarrow GL_n\mathbb{C}$ is an affine variety.

In fact, an <u>algebraic group</u>.

Def A topological space X is <u>irreducible</u> if, whenever $X = X_1 \cup X_2$, X_i closed subsets, then $X = X_1$ or $X = X_2$.

Prop $\mathbb{A}$ vector space U w/ Zariski topology is irreducible.

Pf. Suppose $U = Z(I_1) \cup Z(I_2) = Z(I_1 I_2)$ for ideals I_1, I_2. Then $I_1 I_2 = 0$ since every nonzero polynomial f has a $u \in U$ s.t. $f(u) \neq 0$

$\Rightarrow fg = 0 \quad \forall f \in I_1, \forall g \in I_2$

$\Rightarrow$ ① If $I_1 \neq 0$, then $I_2 = 0$ since $fg \neq 0$ whenever $f \neq 0$ & $g \neq 0$.

② Similarly, if $I_2 \neq 0$, then $I_1 = 0$.

In case 1, $Z(I_2) = U$ ✓

In case 2, $Z(I_1) = U$ ✓

Else, both $I_1 = I_2 = 0$ ✓. $\square$

Prop. Every nonempty open subset Y of an irreducible space is dense (i.e., if $X' \supset Y$ is closed, then $X' = X$)

Pf $X = X' \cup (X \setminus Y)$, both closed.

X irred $\Rightarrow X = X'$ ✓

or $X = X \setminus Y \Rightarrow Y$ is empty $\rightarrow\leftarrow$ $\square$

Now consider U is $GL(V)$-rep., $X \subset U$ affine variety. $G \subset GL(V)$ subgroup s.t. X is closed under action of G.

The G-orbits of X are the equivalence classes of the relation $x \sim x'$ if $\exists g \in G$ s.t. $x' = gx$.

<u>Thm</u> Let X be an affine variety in a rep of $GL(V)$. Let $B \subset GL(V)$ Borel subgroup.

Assume: $\exists$ B-orbit $Y \subset X$ which is dense. <u>Then</u>

(a) $\mathbb{C}[X]$ is multiplicity-free rep.

(b) Let λ be a h.w. of h.w. vector in $\mathbb{C}[X]$.

Pick $u \in Y$ and let $H = \mathrm{stab}(u)$

$$= \{g \in GL(V) \mid h \cdot u = u\}$$

Then, $\lambda(h) = 1$ $\forall$ $h \in H \cap B$.