Math 251C, Spin groups

Note Title 5/28/2020

$C(V)$ has a linear map $x \to x^*$ defined by $(v_1 \cdots v_d)^* = (-1)^d v_d \cdots v_1$ $(v_1, \ldots v_d \in V)$

$*$ is an anti-involution: $(x^*)^* = x$

$$(xy)^* = y^* x^*$$

Define $C(V)^{op}$ to be opposite algebra of $C(V)$:

$$\alpha \cdot \beta = \beta \alpha$$

mult. in $C(V)^{op}$ multiplication in $C(V)$

$*: V \to C(V)^{op}$ universal property:

$$v^* \cdot w^* + w^* \cdot v^* = (-w)(-v) + (-v)(-w) = 2\omega(v,w)$$

Clifford relation holds: get homomorphism

$$*: C(V) \to C(V)^{op}.$$

Also, define $\alpha: C(V) \to C(V)$ by

$$\alpha(v_1 \ldots v_d) = (-1)^d v_1 \cdots v_d$$

$$\mathrm{Pin}(V) = \{ x \in C(V) \mid x x^* = 1,\ x v x^* \in V \ \forall v \in V \}$$

pin group

$$\mathrm{Spin}(V) = \{ x \in C^+(V) \mid x x^* = 1,\ x v x^* \in V \ \forall v \in V \}$$

spin group

$\Longrightarrow$ $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ are groups under mult.

· If we identify $C(V)$ w/ matrix algebra, can see that conditions $xx^* = 1$, $xvx^* \in V$ are given by polynomial equations in entries of matrix.

$\Longrightarrow$ $\mathrm{Pin}(V)$, $\mathrm{Spin}(V)$ are algebraic groups

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$\mathrm{Pin}(V) \longrightarrow \mathbb{Z}/2$ homomorphism

$$x \longmapsto \begin{cases} 0 & \text{if } x \in C^+(V) \\ 1 & \text{if } x \in C^-(V) \end{cases}$$

Kernel is $\mathrm{Spin}(V)$

(we will see that $\mathrm{Pin}(V) \cap C^-(V) \neq \emptyset$, so

$$\mathrm{Spin}(V) \subsetneq \mathrm{Pin}(V))$$

Given $x \in Pin(V)$, define $\rho_x \in GL(V)$ by

$$\rho_x(v) = \alpha(x) v x^*$$

$$= (-1)^{\deg(x)} x v x^* \in V$$

$$\rho_x \rho_y(v) = \alpha(x)\alpha(y) v y^* x^*$$

$$= \alpha(xy) v (xy)^* = \rho_{xy}(v)$$

$\rho : Pin(V) \to GL(V)$ homomorphism.

Thm. $\rho(Pin(V)) = O(V)$, $\ker \rho = \{\pm 1\}$,

$$\rho(Spin(V)) = SO(V).$$

Pf. 1 Show $\rho_x \in O(V)$ $\forall x \in Pin(V)$.

$$2\omega(\rho_x(v), \rho_x(w)) = \rho_x(v)\rho_x(w) + \rho_x(w)\rho_x(v).$$

(if $v \in V$, then $v^* = -v$)

$$= -\rho_x(v)\rho_x(w)^* - \rho_x(w)\rho_x(v)^*$$

$$= -(\alpha(x) v \cancel{x^*})(\cancel{x} w^* \alpha(x)^*) - (\alpha(x) w \cancel{x^*})(\cancel{x} v^* \alpha(x)^*)$$

$$= -\alpha(x) v w^* \alpha(x)^* - \alpha(x) w v^* \alpha(x)^*$$

$$= -\alpha(x)\left(vw^* + wv^*\right)\alpha(x)^*$$

$$= \alpha(x)\,(vw + wv)\,\alpha(x)^*$$

$$= \alpha(x)\left(2\omega(v,w)\right)\alpha(x)^*$$

$$= 2\omega(v,w)\,\alpha(x)\,\alpha(x)^*$$

$$= 2\omega(v,w)\left((-1)^{\deg(x)}\right)^2 x x^* = 2\omega(v,w).$$

$$\Longrightarrow \quad \rho x \in O(V).$$

(2) ρ is surjective onto $O(V)$.

By previous hw, $O(V)$ is generated by reflections s_a ($a \in V$, $\omega(a,a) \neq 0$) where

$$s_a(v) = v - \frac{2\omega(v,a)}{\omega(a,a)}\,a$$

<u>Note</u>: $s_a = s_{\lambda a}$ for $\lambda \in \mathbb{C} \setminus 0$, suffices to use a s.t. $\omega(a,a) = -1$.

In $C(V)$, $aa^* = -a^2 = -\omega(a,a) = 1$

If $x \in V$, $axa^* = (2\omega(a,x) - xa)\,a^*$

$$= -2\omega(a,x)a - x \in V$$

$\Longrightarrow a \in Pin(V)$ (note: $a \in C^-(V)$, so $a \in Pin(V) \setminus Spin(V)$)

Claim: $\rho a = s_a$.

$$\rho_a(v) = \alpha(a) v a^* = -ava^*$$

$$= -(2\omega(a,v) - va)\,a^*$$

$$= 2\omega(a,v)a + v = s_a(v)$$

$\Longrightarrow \rho$ is surjective onto $O(V)$.

③ Show that $\ker \rho \subset \pm 1$.

Pick $x \in \ker \rho$. Write $x = x_0 + x_1$ where $x_0 \in C^+(V)$, $x_1 \in C^-(V)$. Since $\rho_x = id$,

$v = \alpha(x) v x^*$ for all $v \in V$.

$\Longleftrightarrow vx = \alpha(x) v \quad \forall v \in V$.

$\Longrightarrow v(x_0 + x_1) = (x_0 - x_1)\,v \Longrightarrow vx_0 = x_0 v$, $vx_1 = -x_1 v$.

By previous result, $\Rightarrow X_0$ is a scalar

$\Rightarrow X_1 = 0$.

$$1 = x_0^* x_0 = x_0^2 \Rightarrow x_0 = \pm 1.$$

Hence, if $g \in O(V)$, write $g = s_{a_1} \cdots s_{a_r}$ as product of reflections, then $\rho^{-1}(g) = \{a_1 \cdots a_r, -a_1 \cdots a_r\}$

Since $\det(s_a) = -1$, $SO(V)$ is generated by products of even # of reflections, so

$\rho^{-1}(SO(V)) \subseteq Spin(V)$

$\Longrightarrow \rho(Spin(V)) = SO(V)$. $\square$

This gives double cover

$$\begin{array}{ccc} Pin(V) & & Spin(V) \\ \downarrow 2:1 & & \downarrow 2:1 \\ O(V) & & SO(V) \end{array}$$

We know $SO(V)$ connected.

Is same true for $Spin(V)$?

Prop. Spin(V) is connected. if $\dim V \geq 2$

Pf. Euclidean topology refines Zariski topology. hence suffices to check Spin(V) connected in Euclidean topology.

Since $\rho(\mathrm{Spin}(V)) = SO(V)$ is connected, it suffices to show that any 2 elements in $\rho^{-1}(g)$ for $g \in SO(V)$ are connected by a path.

By translation, it suffices to check $g = 1$, i.e., need to check $\pm 1 \in \mathrm{Spin}(V)$ connected by a path.

(Pick $v, w \in V$ s.t. $\omega(v,v) = -1 = \omega(w,w)$ & $\omega(v,w) = 0$)

Define

$$\begin{aligned} [0, \tfrac{\pi}{2}] &\longrightarrow \mathrm{Spin}(V) \\ t &\longmapsto (\cos(t) v + \sin(t) w) \cdot (\cos(t) v - \sin(t) w) \end{aligned}$$

① This is well-defined, i.e., image is in $Spin(V)$

② $0 \longrightarrow vv = v^2 = \omega(v,v) = -1$

$\frac{\pi}{2} \longrightarrow w(-w) = -w^2 = -\omega(w,w) = 1$ $\square$

Rmk. This implies $Pin(V)$ has 2 connected components.

$Pin(V) \subset C(V)$ acts on $\dot{\wedge} W$

$\Rightarrow$ $\dot{\wedge} W$ gives representation of $Pin(V)$

Thm ① $\dot{\wedge} W$ is an irred. rep of $Pin(V)$

② If $\dim V$ is odd, then $\dot{\wedge} W$ is also irred. for $Spin(V)$.

③ If $\dim V$ is even, then $\dot{\wedge} W$ is reducible as $Spin(V)$-rep.

$$\dot{\wedge} W = \overset{\text{even}}{\wedge} W \oplus \overset{\text{odd}}{\wedge} W$$

both irreducible for $Spin(V)$. and non-isom.

Thm ① $\Lambda^{\bullet} W$ is an irred. rep of $Pin(V)$

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③ If $\dim V$ is even, then $\Lambda^{\bullet} W$ is reducible as $Spin(V)$-rep.

$$\Lambda^{\bullet} W = \Lambda^{even} W \oplus \Lambda^{odd} W$$

both irreducible for $Spin(V)$, and non-isom.

$\Lambda^{\bullet} W$ = spinor representation

If $\dim V$ even, $\Lambda^{even} W$, $\Lambda^{odd} W$ = half-spinor representations.

Rmk. Given rep. ρ of $Spin(V)$, it is also rep of $SO(V)$ $\Leftrightarrow$ $\rho(-1) = id$.

Pf of Thm. First, note that linear span of $Pin(V)$ in $C(V)$ is all of $C(V)$: pick a basis $x_1, \dots, x_m$ of V s.t. $\omega(x_i, x_i) = -1$

then $x_I \in Pin(V)$ and form basis for $C(V)$. Since $\bigwedge^\bullet W$ is irred. rep of $C(V)$, same will be true for $Pin(V)$. $\Rightarrow$ (1)

(2) $C^+(V) \cong \mathrm{End}(\bigwedge^\bullet W)$ if $\dim V$ odd, and $Spin(V)$ linearly spans $C^+(V)$.

(3) $C^+(V) \cong \mathrm{End}(\bigwedge^{even} W) \times \mathrm{End}(\bigwedge^{odd} W)$. $\square$

<u>Rmk.</u> If $\dim V = 2n+1$, then h.w. of $\bigwedge^\bullet W$ is ω_n (from $SO(V)$, only got h.w. of the form $\sum f(i)\omega_i$ where $f(n)$ even)

$S_{[\lambda]}V \otimes \bigwedge^\bullet W \ni$ h.w. vector $\otimes$ h.w. vector is h.w. of weight $\lambda + \omega_n$

Can do something similar for $\dim V = 2n$:

Consider $S_{[\lambda]}V \otimes \Lambda^{\text{even}} W$

& $S_{[\lambda]}V \otimes \Lambda^{\text{odd}} W$

$\Rightarrow$ gives all irred. reps for orthogonal Lie algebra that don't come from $SO(V)$