Math 251C, Lecture 25

Note Title 5/27/2020

Thm (FFT for SO, O) Let $\dim V = m$.

① $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{SO(V)}$ is gen. as $\mathbb{C}$-algebra by φ_{ij} and $m \times m$ determinants (if $m \leq n$)

② $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{O(V)}$ is gen. as $\mathbb{C}$-algebra by φ_{ij}.

Pf. R = ring gen by φ_{ij} & $m \times m$ determinants

Last time showed ① $\Rightarrow$ ②

Also can assume $n = m-1$ to prove ①

Now, prove ① by induction on m.

Base case $m=1$: nothing to show.

Now assume $m > 1$. Pick $u \in V$ s.t. $\omega(u,u) = 1$

$V' = \{u\}^{\perp}$, so $V = \langle u \rangle \oplus V'$

If $g \in O(V')$, can extend to element $\tilde{g} \in SO(V)$ by $\tilde{g}(u) = \det g \cdot u$.

If $f \in \mathrm{Sym}(V \otimes \mathbb{C}^n)^{SO(V)}$, then restriction $f' \in \mathrm{Sym}(V' \otimes \mathbb{C}^n)^{O(V')}$

By induction, $f' =$ polynomial in q_{ij}.

Let F be same polynomial in q_{ij} inside $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{SO(V)}$.

If u' is any other vector of norm 1, then $\exists g \in SO(V)$ s.t. $g(u) = u'$ & hence $F - f$ restricted to $\mathrm{Sym}(\langle u' \rangle^\perp \otimes \mathbb{C}^n)$ is 0. The set of vectors of nonzero norm is Zariski dense in $V \Longrightarrow F - f$ is 0 on this set $\Longrightarrow F = f$. $\square$

$V^{\otimes n} \circlearrowleft O(V)$.

If $n=2k$ even, pick a perfect matching M of $\{1,\dots,n\}$, i.e., write $\{1,\dots,n\}$ as disjoint union of 2-element subsets. $\rightsquigarrow$

$\varphi_M : V^{\otimes n} \longrightarrow \mathbb{C}$ linear, $O(V)$-invariant

$$\varphi_M(v_1 \otimes \cdots \otimes v_n) = \prod_{\{i,j\}\in M} \omega(v_i, v_j).$$

$\varphi_M \in (V^{\otimes n})^* \cong V^{\otimes n}$ since ω gives identification $V \cong V^*$.

<u>Cor.</u> $(V^{\otimes n})^{O(V)} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \text{spanned by } \varphi_M & \text{if } n \text{ even} \end{cases}$

Pf. $\mathrm{Sym}(V\otimes\mathbb{C}^n) = \mathrm{Sym}(V^{\oplus n}) = \mathrm{Sym}(V)^{\otimes n}$

Contains $V^{\otimes n}$ as a linear subspace, namely the $(1,\dots,1)$-weight space under the maximal torus in $GL_n(\mathbb{C})$.

FFT $\Longrightarrow$ $O(V)$-invariants of $\mathrm{Sym}(V\otimes\mathbb{C}^n)$ gen. by φ_{ij}. $\longrightarrow$ weight vector of weight

$$(0 \dots \underset{i}{1}, \dots \underset{j}{1} \dots 0)$$

$\Longrightarrow$ $(V^{\otimes n})^{O(V)}$ spanned by products of φ_{ij} w/ total weight $(1 \dots 1)$.

i.e., φ_M, over all perfect matchings M of $\{1,\dots,n\}$. $\square$

Now consider $G = Sp(V)$, V symplectic w/ form ω. For $1 \le i < j \le n$, have $\varphi_{ij} \in \mathrm{Sym}(V \otimes \mathbb{C}^n)$ given by

$$\varphi_{ij}\left(\sum v_i \otimes e_i \right) = \omega(v_i, v_j).$$

If $i > j$, define $\varphi_{ij} = -\varphi_{ji}$, $\varphi_{ii} = 0$.

<u>Thm</u> (FFT for $Sp(V)$). $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{Sp(V)}$ is generated as $\mathbb{C}$-algebra by φ_{ij}.

<u>Pf</u>. R = subalgebra gen. by φ_{ij}. R closed under $GL_n\mathbb{C}$. $\Rightarrow$ suffices to consider $n = \dim V$, assume that.

Let u be generic matrix. Then

$$u^T \Omega u = (\varphi_{ij}) = \Psi \leftarrow \text{skew-symmetric}$$

$$\mathrm{Pf}(\Psi) = \mathrm{Pf}(\Omega)\det(u) = \pm \det(u)$$

$\mathrm{Pf}(\Psi)$ is polynomial in $\psi_{ij} \Rightarrow \mathrm{Pf}(\psi) \in R$

$\Longrightarrow \det(u) \in R$, so R already contains

$m \times m$ determinants.

$\Rightarrow$ can assume $m = \dim V - 1$.

Idea: do induction on $\dim V$.

Induction step similar to orthogonal case.

Details in notes. $\square$

$V^{\otimes n} \circlearrowleft Sp(V)$. If $n = 2k$ even, pick perfect matching M of $\{1, \dots n\}$, define

$$\psi_M : V^{\otimes n} \longrightarrow \mathbb{C}$$

$$\psi_M(v_1 \otimes \dots \otimes v_n) = \prod_{\{i<j\} \in M} \omega(v_i, v_j)$$

As before:

<u>Cor.</u> $(V^{\otimes n})^{Sp(V)} = \begin{cases} 0 & \text{if } n \text{ odd} \\ \text{spanned by } \psi_M & \text{if } n \text{ even.} \end{cases}$

Brauer algebras.

Goal: compute commutant of $G \curvearrowright V^{\otimes n}$ where $G = O(V)$ or $G = Sp(V)$.

$$\mathrm{End}(V^{\otimes n}) = V^{\otimes n} \otimes (V^{\otimes n})^* \overset{\omega}{\cong} V^{\otimes 2n}$$

Commutant of G is identified w/ $(V^{\otimes 2n})^G$

Given a perfect matching M of $\{1, \ldots, 2n\}$, get $\varphi_M / \psi_M \in (V^{\otimes 2n})^G \rightsquigarrow \beta_M \in \mathrm{End}(V^{\otimes n})$ acts on $v_1 \otimes \cdots \otimes v_n$ as follows:

1. Given edge $\{i, j\}$ w/ $i, j \leq n$, we remove v_i, v_j and scale result $\omega(v_i, v_j)$.
2. Given edge $\{i, j+n\}$ w/ $i, j \leq n$, then move v_i to the j^{th} tensor position.

(The set of edges forms a bijection between a subset of $\{1, \ldots, n\}$ and subset of $\{n+1, \ldots, 2n\}$

In symplectic case, multiply by sign of this bijection.

③ For edges $\{i+n, j+n\}$, $i,j \le n$, we insert element Ω in positions i,j, where $\Omega \in V^{\otimes 2}$ is given by

$$\Omega = \sum_{i=1}^{m} e_i \otimes e_{m+1-i}$$

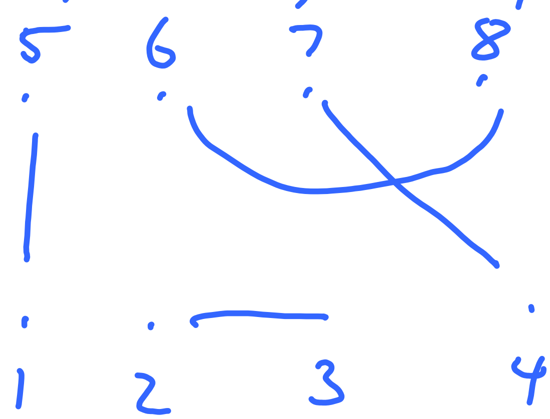
$\{e_1, \ldots, e_m\}$ hyperbolic basis in $O(V)$ case

$$\Omega = \sum_{i=1}^{m} -e_i \otimes e_{2m+1-i} + e_{2m+1-i} \otimes e_i$$

$\{e_1, \ldots e_{2m}\}$ symplectic basis for $Sp(V)$ case.

Ex. $G = O(V)$, $n = 4$.

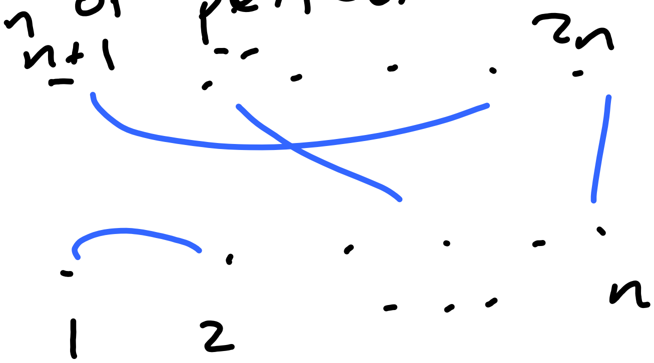
M:



$$\beta_M(v_1 \otimes v_2 \otimes v_3 \otimes v_4) = \omega(v_2, v_3) \sum_{i=1}^{m} v_1 \otimes e_i \otimes v_4 \otimes e_{m+1-i}$$

Thm The commutant $B = A^!$ is linearly spanned by β_M, M ranges over all perfect matchings of $\{1, \dots, 2n\}$.

We first define product structure on the set B_n of perfect matchings of $\{1, \dots 2n\}$.

$M \longleftrightarrow$ 

edges = 2-element subsets.

To multiply M, M':

(1) Stack M' on top of M

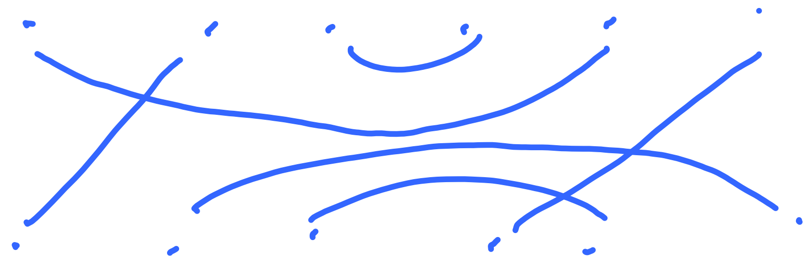
(2) Identify the middle row of vertices

(3) Get induced perfect matching on union of bottom row & top row. Result is MM'.

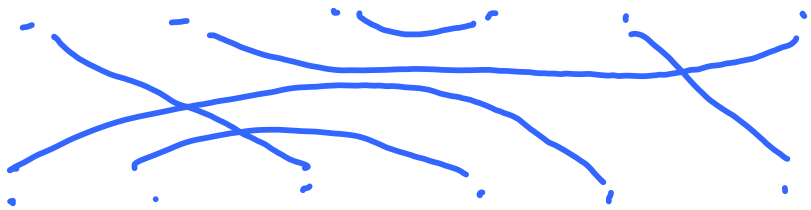
$n(M, M')$ = number of closed loops in middle row (which is discarded).

Ex. $n=6$

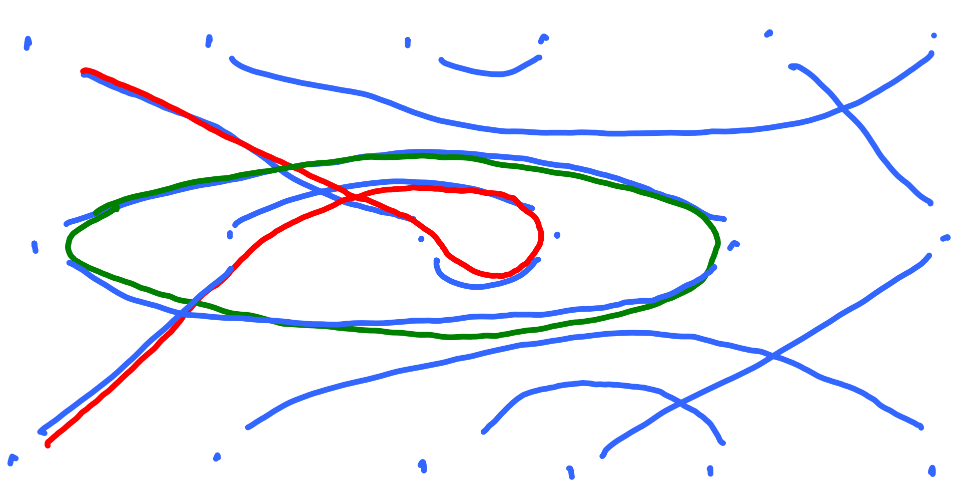
M:



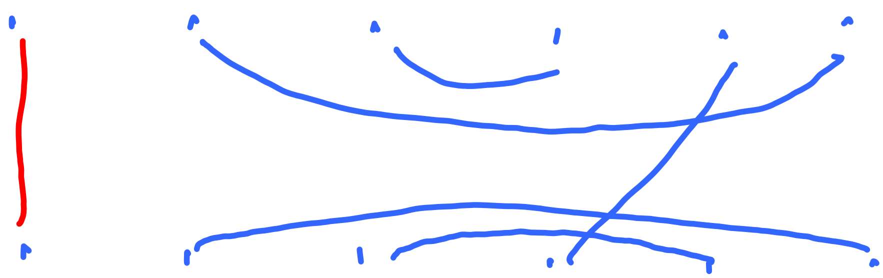
M':



stack them:



MM':



$n(M, M') = 1$

Def Pick $\delta \in \mathbb{C}$. The **Brauer algebra** $B_n(\delta)$ has basis given by perfect matchings e_M. w/ product

$$e_M \cdot e_{M'} = \delta^{n(M,M')} MM'$$

product we just defined

$(0^0 = 1)$

Thm. We have surjective algebra homomorphisms

$$B_n(\dim V) \longrightarrow \mathrm{End}_{O(V)}(V^{\otimes n})$$

$$B_n(-\dim V) \longrightarrow \mathrm{End}_{Sp(V)}(V^{\otimes n})$$

which sends $e_M \longmapsto \beta_M$.

(If $\dim V \geq n$, then, these are isomorphisms)

By double commutant, image is semisimple, but $B_n(\delta)$ need not be semisimple!