Math 251C, Lecture 24

Note Title 5/22/2020

<u>Schur-Weyl duality</u> V = f.d. vector space

$V^{\otimes n}$ is rep of $GL(V)$

also rep of $\mathfrak{S}_n$ symmetric group

$$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$$

Representation structures give $(\mathcal{U} = \mathrm{End}(V^{\otimes n}))$

$GL(V) \longrightarrow \mathcal{U}$, A = linear span of image
$\mathbb{C}$-subalgebra of $\mathcal{U}$

$\mathfrak{S}_n \longrightarrow \mathcal{U}$, B = linear span of image
$\mathbb{C}$-subalg. of $\mathcal{U}$

<u>Def</u> Given subalg. $R \subset \mathrm{End}(W)$, let
$R^! = \{x \in \mathrm{End}(W) \mid xr = rx \ \forall r \in R\}$

$$A \subseteq B^!, \quad B \subseteq A^!$$

<u>Prop</u> $B^! = A$

<u>Pf</u>. $U = \mathrm{End}(V^{\otimes n}) = (\mathrm{End}(V))^{\otimes n}$ as follows:

given $A_1 \otimes \cdots \otimes A_n$, $A_i \in \mathrm{End}(V)$, get

$V^{\otimes n} \longrightarrow V^{\otimes n}$ by $v_1 \otimes \cdots \otimes v_n \longrightarrow A(v_1) \otimes \cdots \otimes A(v_n)$

Under this identification, image of $g \in GL(V)$ in U

goes to $g \otimes g \otimes \cdots \otimes g \in (\mathrm{End}\, V)^{\otimes n}$

$$A = \{ \varphi \in (\mathrm{End}\, V)^{\otimes n} \mid \sigma\varphi = \varphi \ \forall \sigma \in \mathfrak{S}_n \}$$

Need to identity symmetric tensors in $\mathrm{End}(V^{\otimes n})$

Pick $A_1 \otimes \cdots \otimes A_n \in \mathrm{End}(V)^{\otimes n}$, $\sigma \in \mathfrak{S}_n$

$$\sigma(A_1 \otimes \cdots \otimes A_n) = A_{\sigma^{-1}(1)} \otimes \cdots \otimes A_{\sigma^{-1}(n)}$$

in $\mathrm{End}(V^{\otimes n})$, this corresponds to

$$\begin{aligned} v_1 \otimes \cdots \otimes v_n \longrightarrow A_{\sigma^{-1}(1)}(v_1) \otimes \cdots \otimes A_{\sigma^{-1}(n)}(v_n) &\in V^{\otimes n} \\ = \sigma \cdot (A_1(v_{\sigma(1)}) \otimes \cdots \otimes A_n(v_{\sigma(n)})) & \\ = \sigma \circ (A_1 \otimes \cdots \otimes A_n) \circ \sigma^{-1} (v_1 \otimes \cdots \otimes v_n) & \end{aligned}$$

$\Rightarrow \alpha$ being symmetric in $\mathrm{End}(V)^{\otimes n}$ translates to $\sigma \alpha \sigma^{-1} = \alpha \ \forall \sigma \in \mathfrak{S}_n$ in $\mathrm{End}(V^{\otimes n}) \Rightarrow \alpha \in B^!$

Being symmetric in $(\mathrm{End}(V))^{\otimes n}$ is equivalent to being in $B^!$ in $\mathrm{End}(V^{\otimes n})$. $\Rightarrow A = B^!$ $\square$

<u>Cor</u>. $A^! = B$

<u>Pf</u>. B = quotient of group algebra of $\mathfrak{S}_n$

$\mathbb{C}[\mathfrak{S}_n]$ is semisimple $\Rightarrow B$ semisimple

use double commutant thm $\square$

<u>Cor</u>. As $\mathfrak{S}_n \times GL(V)$-representation, get

$$V^{\otimes n} \cong \bigoplus_{\lambda} S^{\lambda} \otimes S_{\lambda}(V)$$

partitions of n s.t. $\ell(\lambda) \leq \dim V$

where S^{λ} = irred. $\mathfrak{S}_n$-rep.

<u>Pf</u>. By Pieri's rule, $V^{\otimes n}$ decomposes as sum of $S_{\lambda}(V)$ where $\ell(\lambda) \leq \dim V$ and $|\lambda| = n$ w/ multiplicity given by sequences

$$\square \subseteq \lambda^{(2)} \subseteq \lambda^{(3)} \subseteq \cdots \subseteq \lambda^{(n)} = \lambda$$

where $\lambda^{(i+1)}/\lambda^{(i)}$ = one box, $\lambda^{(i)}$ partitions.

For every λ, at least one such sequence exists. $\square$

Rmk. S^λ ← Specht modules

dim = mult of $S_\lambda(V)$ = # tableau of shape λ using $1, \dots, n$, each exactly once

(i.e., standard Young tableaux)

Next goal: replace $GL(V)$ by either $O(V)$, $Sp(V)$ that makes A smaller, and necessarily B gets larger (quotients of Brauer algebras)

First fundamental theorem of invariant theory (FFT)

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setup: E, V vector spaces. $\dim V = m$, $\dim E = n$

$G \subset GL(V)$. Consider ring of invariants

$$\mathrm{Sym}(V \otimes E)^G = \{ f \in \mathrm{Sym}(V \otimes E) \,/\, gf = f \ \forall g \in G \}$$

$$\| \qquad \bigoplus_\lambda S_\lambda(V)^G \otimes S_\lambda(E)$$

$GL(E)$ commutes w/ $GL(V)$ (and hence G), so $GL(E)$ acts on $\mathrm{Sym}(V \otimes E)^G$.

<u>Lemma</u>. If $\dim V = m$, & $n \geq m$, then $\mathrm{Sym}(V \otimes \mathbb{C}^n)^G$ is generated by $\mathrm{Sym}(V \otimes \mathbb{C}^m)^G$ together w/ action of $GL_n\mathbb{C}$.

<u>pf</u>. $\mathrm{Sym}(V \otimes \mathbb{C}^n)^G = \bigoplus_{\lambda} S_\lambda(V)^G \otimes S_\lambda(\mathbb{C}^n)$ $\leftarrow$ $\ell(\lambda) \leq \min(n,m) = m$

so h.w. vector for $S_\lambda \mathbb{C}^n$ only uses first m basis vectors. $\Longrightarrow$ $S_\lambda \mathbb{C}^n$ is generated by $S_\lambda \mathbb{C}^m$ together w/ action of $GL_n\mathbb{C}$.

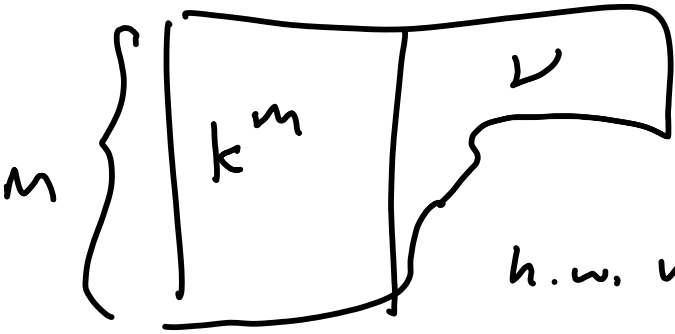
$\Longrightarrow$ $S_\lambda(V)^G \otimes S_\lambda(\mathbb{C}^n)$ is generated by $S_\lambda(V)^G \otimes S_\lambda(\mathbb{C}^m)$ together w/ $GL_n\mathbb{C}$. $\square$

Consider condition $G \subseteq SL(V)$, i.e., $\det g = 1$ $\forall g \in G$.

if we pick bases of V, E. can identify $V \otimes E$ as $n \times m$ matrices. The determinants of $m \times m$ submatrices are invariant under $SL(V)$, hence G.

<u>Lemma</u>. If $G \subseteq SL(V)$, and $n \geq m$, then $\mathrm{Sym}(V \otimes \mathbb{C}^n)^G$ is gen. by $\mathrm{Sym}(V \otimes \mathbb{C}^{m-1})^G$ together w/ $m \times m$ determinants and action of $GL_n\mathbb{C}$.

Pf. When $\lambda = (1^m)$, $S_\lambda V = \Lambda^m V$ is the determinant rep. Furthermore, if $\ell(\mu) = m$, then $\mu = (k^m) + \nu$ where $\ell(\nu) < m$. $(k = \mu_m)$



$\Longrightarrow S_\mu \mathbb{C}^m \cong \det(\mathbb{C}^m)^{\otimes k} \otimes S_\nu \mathbb{C}^m$

h.w. vector for $S_\mu \mathbb{C}^m$ is h.w. vector for $S_\nu \mathbb{C}^m$ times $\det^k$

$\Longrightarrow (S_\mu V)^G \otimes S_\mu \mathbb{C}^m$ is generated by $(S_\nu V)^G \otimes S_\nu \mathbb{C}^m$ by multiplying by $\det^k$. $\square$

Now consider $G = O(V)$ w/ form ω.
ω gives identification $V \cong V^*$: given $v \in V$, get linear functional $u \longrightarrow \omega(u, v)$. This gives G-invariant subspace of $\mathrm{Sym}^2 V$ spanned by ω. $\Longrightarrow$ get G-invt subspace in $\mathrm{Sym}^2 V \otimes \mathrm{Sym}^2 E$ spanned by $\omega \otimes ee'$ $\quad e, e' \in E$.

More explicitly, this is spanned by φ_{ij} $(1 \le i \le j \le n)$ defined by

$$\varphi_{ij}\left(\sum_{k=1}^{n} v_k \otimes e_k\right) = \omega(e_i, e_j)$$

$e_1 \dots e_n$ basis for E

As matrices, φ_{ij} applies ω to i^{th} & j^{th} columns.

If $j < i$, define $\varphi_{ij} = \varphi_{ji}$.

<u>Thm</u> (FFT for SO, O) Let $\dim V = m$.

① $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{SO(V)}$ is gen. as $\mathbb{C}$-algebra by φ_{ij} and $m \times m$ determinants (if $m \le n$)

② $\mathrm{Sym}(V \otimes \mathbb{C}^n)^{O(V)}$ is gen. as $\mathbb{C}$-algebra by φ_{ij}.

<u>Pf</u> Let R_n be $\mathbb{C}$-algebra gen by φ_{ij} and $m \times m$ determinants. <u>Note</u>: R_n is closed under action of $GL_n\mathbb{C}$ (since ω is bilinear)

Step 1. Show that ① $\Rightarrow$ ②. Suffices to assume $m=n$ ($GL_n\mathbb{C}\cdot R_m = R_n$)

$g\in O(V)$ acts on $m\times m$ determinants by ± 1

So, assuming ① holds, $\mathrm{Sym}(V\otimes\mathbb{C}^n)^{O(V)}$ is gen. by φ_{ij} and products of even # of $m\times m$ determinants

If u is the generic $m\times n$ matrix, we have

$$u^T I' u = (\varphi_{ij}) = \Phi$$

$$\Rightarrow \det\Phi = \det(u)^2\cdot(-1)^m$$

$\det\Phi$ gen by $\varphi_{ij} \Rightarrow \det(u)^2$ gen by φ_{ij}

$\Rightarrow (\det u)^{2k}$ gen by φ_{ij}

Hence ① $\Rightarrow$ ②.