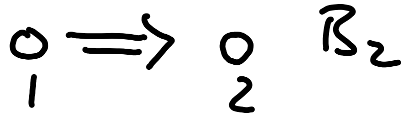
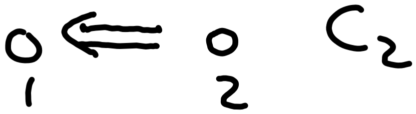
Math 251C, Lecture 23

Note Title 5/20/2020

Accidental Isomorphisms

Ex 1. $B_2 = C_2$



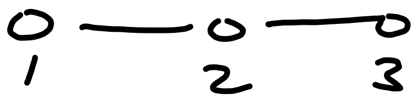


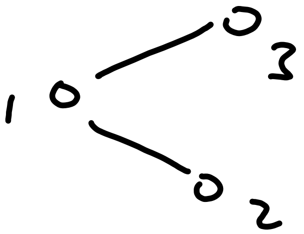
Consequences:

- $\mathfrak{so}_5\mathbb{C} \cong \mathfrak{sp}_4\mathbb{C}$
- $\mathrm{Spin}_5\mathbb{C} \cong \mathrm{Sp}_4\mathbb{C}$ ($\exists\ \mathrm{Sp}_4\mathbb{C} \to \mathrm{SO}_5\mathbb{C}$ w/ kernel (of size 2))
- $\underset{\underset{\mathrm{SO}_5\mathbb{C}}{\circlearrowright}}{\mathbb{C}^5} \cong \underset{\underset{\mathrm{Sp}_4\mathbb{C}}{\circlearrowright}}{\Lambda^2\mathbb{C}^4/\langle\Omega\rangle} \Rightarrow \Lambda^2\mathbb{C}^4/\langle\Omega\rangle$ has an orthogonal form.
- Spin rep $\cong \mathbb{C}^4$
- $\underset{\text{quadric in } \mathbb{P}^4}{\underset{\wr\|}{OF\ell(1;5)}} \cong IF\ell(2;4)$ $\Rightarrow IF\ell(2;4)$ also quadric in $\mathbb{P}^4$

Pfaffian of $\Lambda^2\mathbb{C}^4$ restricted to complement of Ω.

Ex 2. $A_3 = D_3$





Consequences:

- $\mathfrak{sl}_4\mathbb{C} \cong \mathfrak{so}_6\mathbb{C}$
- $SL_4\mathbb{C} \cong \mathrm{Spin}_6\mathbb{C}$ ($\exists\ SL_4\mathbb{C} \to SO_6\mathbb{C}$ w/ kernel of size 2)
- $\underset{\underset{SL_4\mathbb{C}}{\circlearrowleft}}{\Lambda^2\mathbb{C}^4} \cong \underset{\underset{\mathfrak{so}_6\mathbb{C}}{\circlearrowleft}}{\mathbb{C}^6} \implies \Lambda^2\mathbb{C}^4$ has an orthogonal form

given $\alpha, \beta \in \Lambda^2\mathbb{C}^4$, $\omega(\alpha,\beta) =$ coeff. of $e_1\wedge e_2\wedge e_3\wedge e_4$ in $\alpha\wedge\beta$

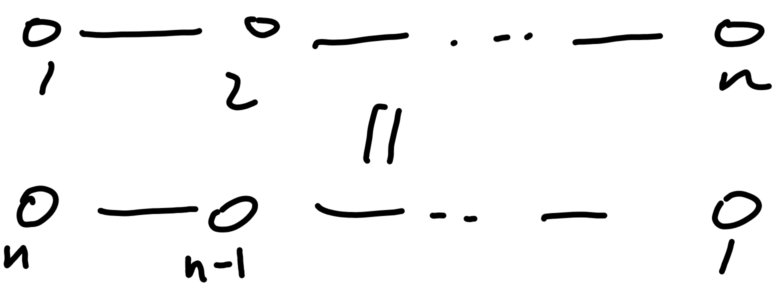
- $Gr(2,4) \cong OF\ell(1;6)$

  $\wr\wr$ $\underline{\text{quadric}}$ in $\mathbb{P}^5$

  Pfaffian

- $\mathbb{P}^3 \cong OF\ell(3;6)$; $Gr(3,4) \cong OF\ell(3';6)$
- $F\ell(1,3;4) \cong OF\ell(3,3';6) \cong OF\ell(2;6)$

EX 3. $A_n = A_n$ via nontrivial automorphism



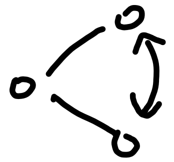
- $\underset{\underset{SL_{n+1}\mathbb{C}}{\circlearrowright}}{\bigwedge^i \mathbb{C}^{n+1}} \cong \underset{\underset{SL_{n+1}\mathbb{C}}{\circlearrowright}}{\bigwedge^{n+1-i} (\mathbb{C}^{n+1})^*}$

b/c isomorphism $SL_{n+1}\mathbb{C} \cong SL_{n+1}\mathbb{C}$ via $g \to (g^{-1})^T$

- $Gr(i, n+1) \cong Gr(n+1-i, n+1)$.'

  $W \longrightarrow (\mathbb{C}^{n+1}/W)^*$

D_n also has a nontrivial automorphism.



If n is odd, this swaps reps for their duals

Classical invariant theory

Schur-Weyl duality $V =$ f.d. vector space

$V^{\otimes n}$ is rep of $GL(V)$

also rep of $\mathfrak{S}_n$ symmetric group

$\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(n)}$

Define $End(W) = Hom(W, W)$ for any vector space W

$\mathcal{U} = \mathrm{End}(V^{\otimes n})$ is a $\mathbb{C}$-algebra w/ composition as multiplication.

Representation structures give

$GL(V) \longrightarrow \mathcal{U}$, A = linear span of image
$\mathbb{C}$-subalgebra of $\mathcal{U}$

$\mathfrak{S}_n \longrightarrow \mathcal{U}$, B = linear span of image
$\mathbb{C}$-subalg. of $\mathcal{U}$

Def Given subalg. $R \subset \mathrm{End}(W)$, let
$R' = \{x \in \mathrm{End}(W) \mid xr = rx \ \forall r \in R\}$ commutant

$A \subseteq B'$, $B \subseteq A'$

Def. A f.dim $\mathbb{C}$-algebra is semisimple if it is isomorphic to direct product of matrix algebras.

Thm (Double commutant theorem) Let $R \subset \mathrm{End}(W)$ be semisimple subalg. Then:

① $S := R'$ is semisimple, and $R = S'$

② As $R \times S$-rep, $W \cong \bigoplus_{i \in I} M_i \otimes N_i$ where

M_i = simple R-reps & $M_i \not\cong M_{i'}$ for $i \neq i'$.
N_i = simple S-reps $\quad N_i \not\cong N_{i'}$

Pf. R semisimple $\Longrightarrow W \cong \bigoplus_{i\in I} M_i^{\oplus m_i}$

where M_i = simple, distinct R-reps. &

$R = \prod_{i\in I} \mathrm{End}(M_i)$. By Schur's lemma,

$S = R^! = \prod_{i\in I} \mathrm{End}(\mathbb{C}^{m_i}) \Longrightarrow$ semisimple.

$\Longrightarrow$ As S-rep, $W \cong \bigoplus_{i\in I} N_i^{\oplus \dim M_i}$ where $N_i = \mathbb{C}^{m_i}$.

By Schur's lemma again, $S^! \cong \prod_{i\in I} \mathrm{End}(\mathbb{C}^{\dim M_i})$

But $R \subseteq S^!$ and $\dim R = \dim S^! \Longrightarrow R = S^!$ $\square$

Prop $B^! = A$

Pf. $U = \mathrm{End}(V^{\otimes n}) = (\mathrm{End}(V))^{\otimes n}$ as follows:

given $A_1 \otimes \cdots \otimes A_n$, $A_i \in \mathrm{End}(V)$, get

$V^{\otimes n} \longrightarrow V^{\otimes n}$ by $v_1 \otimes \cdots \otimes v_n \longrightarrow A(v_1) \otimes \cdots \otimes A(v_n)$

Under this identification, image of $g \in GL(V)$ in U goes to $g \otimes g \otimes \cdots \otimes g \in (\mathrm{End}\, V)^{\otimes n}$

Claim. $A = \{\underbrace{\varphi \in (\mathrm{End}\,V)^{\otimes n} \mid \sigma\varphi = \varphi \ \forall \sigma \in \mathfrak{S}_n}_{\text{symmetric tensors}}\}$

Pf of claim: Pick f linear fcn on symmetric tensors.

$\Longrightarrow$ degree n polynomial F on $GL(V)$ via

$F(g) = f(g \otimes \cdots \otimes g)$, in fact F is defined on $\mathrm{End}(V)$

$GL(V) \subset \mathrm{End}(V)$ is Zariski dense $\Longrightarrow$ F is identically 0 on $GL(V) \Longleftrightarrow F \equiv 0$ on $\mathrm{End}(V)$.

In particular, f is identically 0 on A

$\Longrightarrow F \equiv 0$ on $GL(V)$

$\Longrightarrow F \equiv 0$ on $\mathrm{End}(V)$

$\Longrightarrow f = 0$ on symmetric tensors.

(symmetric tensors spanned by $\{X \otimes \cdots \otimes X \mid X \in \mathrm{End}(V)\}$) $\square$