Math 251C, Lecture 20

Note Title 5/13/2020

$\mathcal{L}_\lambda := \mathrm{Ind}_B^G \mathbb{C}_\lambda$. line bundle on G/B (i.e., rank 1 vector bundle).

<u>Thm</u> (Borel-Weil). If λ dominant, i.e., $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, then $H^0(G/B; \mathcal{L}_\lambda)^* \cong S_\lambda \mathbb{C}^n$.

<u>Pf</u>. Last time: $H^0(G/B; \mathcal{L}_\lambda)^*$ irreducible (or 0).

Now: construct L-fixed section $s: G/B \to G \times \mathbb{C}_\lambda / \sim$

The data of s is the same as a function $f: G \to \mathbb{C}$ s.t. $f(gb) = \lambda(b) f(g) \quad \forall b \in B, \forall g \in G$.

Let $f_i(g) =$ det of upper left $i \times i$ submatrix of g.

$$f(g) = \prod_{i=1}^n f_i(g)^{\underbrace{\lambda_i - \lambda_{i+1}}} \quad (\lambda_{n+1} = 0)$$

≥ 0 for $i \leq n-1$

for $i = n$, $f_n(g) \neq 0$

$\Rightarrow$ well-defined.

$$\Rightarrow f(gb) = f(g) \prod_{i=1}^n f_i(b)^{\lambda_i - \lambda_{i+1}} = f(g)\lambda(b)$$

$\Rightarrow$ We have section s.

- s is fixed under L: $\forall \ell \in L$, $f'(\ell) = 1$.
- Pick t diagonal. Then

$$(t \cdot s)(gB) = t(s(t^{-1}gB)) = t(t^{-1}g, f(t^{-1})f(g))$$
$$= t(t^{-1}g, \lambda(t^{-1})f(g))$$
$$= (g, \lambda(t^{-1})f(g)).$$

$$s(gB) = (g, f(g))$$

weight of s is $-\lambda$.

$\Rightarrow$ corresponding h.w. in dual space has weight λ. $\square$

<u>Rmk</u>. As B-reps, $\mathbb{C}_\lambda \otimes \mathbb{C}_\mu = \mathbb{C}_{\lambda+\mu}$.

$\Rightarrow \mathcal{L}_\lambda \otimes \mathcal{L}_\mu \cong \mathcal{L}_{\lambda+\mu}$. as homog. bundles.

For any X, $\mathrm{Pic}(X) = \{\text{line bundles on } X\}/\text{isom.}$ Picard group.

is a group under tensor product w/ $\mathcal{L}^{-1} = \mathcal{L}^*$.

For $X = G/B$, $\mathcal{L}_{(1^n)}$ is trivial: it has a section which is never 0: $gB \longrightarrow (g, \det g)$.

$$G/B \times \mathbb{C} \longrightarrow \mathcal{L}_{(1^n)}$$
$$(gB, \alpha) \longmapsto (g, \alpha \det g)/\sim$$

$\Longrightarrow \mathrm{Pic}\, G/B \cong \mathbb{Z}^n/\mathbb{Z}(1, \dots, 1) \cong \mathbb{Z}^{n-1}$.

<u>Rmk</u>: Relative Flag varieties:

given a vector bundle E on a variety X,
$\exists$ variety $Fl(\underline{d}; E)$ together w/ map $\pi: Fl(\underline{d}; E) \to X$
s.t. $\pi^{-1}(x) = Fl(\underline{d}; E_x)$.

Taking sections $\rightsquigarrow$ pushforward

Relative Borel-weil: $\pi_* \mathcal{L}_\lambda \cong (S_\lambda E)^*$ when $\underline{d} = (1, \ldots, n-1)$

<u>Rmk</u>. H^0 notation suggests we have H^i for $i \in \mathbb{Z}$.

Idea: in general, given $F \subset E$ vector bundles, we have maps $H^0(X; F) \to H^0(X; E) \to H^0(X; E/F)$

inclusion always

composition is 0
kernel of second map is image of first

Second map need not be surjective.

Long exact sequence (image of map = kernel of next)

$$\cdots \longrightarrow H^0(X; E/F) \to H^1(X; F) \to H^1(X; E) \to \cdots$$

$H^i(G/B; \mathcal{L}_\lambda)$ is computed by Borel-Weil-Bott.

for each λ, $H^i \neq 0$ for at most one i.

Symplectic/orthogonal flag varieties

Let $\mathbb{C}^{2n}$ = symplectic space w/ form ω

$\underline{d} = (d_1 \cdots d_r)$ $1 \le d_1 < d_2 < \cdots < d_r \le n$

$IF\ell(\underline{d};\mathbb{C}^{2n}) = \{ W_{d_1} \subset W_{d_2} \subset \cdots W_{d_r} \subseteq \mathbb{C}^n \mid$ each W_{d_i} isotropic, $\dim W_{d_i} = d_i \}$

$\Rightarrow Sp_{2n}\mathbb{C}$ acts transitively on $IF\ell(\underline{d};\mathbb{C}^{2n})$

symplectic flag variety.

$IF\ell(2n) = IF\ell(1, 2, \ldots, n; 2n)$.

<u>Prop.</u> $IF\ell(\underline{d};2n)$ is a projective variety.

<u>Pf.</u> $IF\ell(\underline{d};2n) \subseteq F\ell(\underline{d};2n)$. As a subset, it is the flags s.t. W_{d_r} isotropic.

$R_{d_r} \subset \mathbb{C}^{2n} \times F\ell(\underline{d};2n)$ tautological subbundle.

Define $s: F\ell(\underline{d};2n) \to (\bigwedge^2 R_{d_r})^*$ as follows:

for $W_\bullet \in F\ell(\underline{d};2n)$, define $s(W_\bullet): \bigwedge^2 W_{d_r} \to \mathbb{C}$

$$u \wedge v \longmapsto \omega(u,v)$$

$IF\ell(\underline{d};2n) = s^{-1}(IF\ell(\underline{d};2n)_0)$, hence closed. $\square$

Rmk. Given a vector bundle E on space X, E is symplectic if has linear map $\bigwedge^2 E \to \mathcal{L}$, $\mathcal{L}$ line bundle on X s.t. $\forall x \in X$, $\bigwedge^2 E_x \to \mathbb{C} = \mathcal{L}_x$ is symplectic form.

Can define $\pi : IF\ell(\underline{d}; E) \to X$ s.t.

$$\pi^{-1}(x) = IF\ell(\underline{d}; E_x) \quad \forall x \in X.$$

If $\underline{d} = (1)$, $IF\ell(\underline{d}; 2n) = \mathbb{P}^{2n-1}$.

since all 1-dim subspaces are isotropic.

Prop. $\dim IF\ell(2n) = n^2$.

Pf. R = tautological subbundle on $\mathbb{P}^{2n-1}$.

$R^\perp \subset \mathbb{C}^{2n} \times \mathbb{P}^{2n-1}$ is $\left\{(v, W) \;\middle|\; \begin{array}{l}\omega(v,u) = 0 \\ \forall u \in W\end{array}\right\}$

$\Rightarrow R \subset R^\perp$ and $R^\perp / R$ symplectic

$IF\ell(2n) = IF\ell(R^\perp / R)$.

$\Rightarrow \dim IF\ell(2n) = \dim \mathbb{P}^{2n-1} + \dim IF\ell(2n-2)$

Finish by induction. $\square$

Rmk. Get surjective maps $IFl(\underline{d};2n) \to IFl(\underline{d}';n)$ whenever $\underline{d}'$ is subsequence of $\underline{d}$. fibers are products of GL-flag varieties. Can use this to compute dim in general case

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Orthogonal case:

$\mathbb{C}^{2n+1}$ orthogonal space

$\underline{d} = (d_1, \ldots d_r)$ $1 \le d_1 < \cdots < d_r \le n$.

$$OFl(\underline{d};2n+1) = \{ W_{d_1} < \cdots < W_{d_r} < \mathbb{C}^{2n+1} \mid W_{d_i} \text{ isotropic}, \dim W_{d_i} = d_i \}$$

$SO_{2n+1}\mathbb{C}$ acts transitively on $OFl(\underline{d};2n+1)$

projective variety ✓

$\mathbb{C}^{2n}$ orthogonal space

Recall: $SO_{2n}\mathbb{C}$ has 2 orbits on n-dim isotropic subspaces of $\mathbb{C}^{2n}$. Can distinguish by parity of $\dim(W \cap \operatorname{span}\{e_1, \ldots, e_n\})$. We call W even/odd depending on this parity

Lemma. Given $(n-1)$-dim'l isotropic subspace W of $\mathbb{C}^{2n}$, $\exists$ exactly 2 n-dim'l isotropic subspaces containing W.

Pf. $\left\{\begin{array}{c} n\text{-dim'l isotropic subspaces} \\ \text{containg } W \end{array}\right\}$

$$\updownarrow$$

$\left\{\begin{array}{c} 1\text{-dim'l isotropic subspaces} \\ \text{of } W^{\perp}/W \end{array}\right\}$

$\dim W^{\perp}/W = 2$, and the form in some is given by $\omega(ae_1+be_2, ce_1+de_2)$
$= ad+bc$

ae_1+be_2 is isotropic $\iff ab=0$

2 solutions up to scalar multiple. $\square$