Math 251C, Lecture 2

Note Title 4/1/2020

Basic operations

① Direct sum $\rightsquigarrow$ sum of characters

② Tensor products:

Given vector spaces U, V. The tensor product $U \otimes V$ is the vector space spanned by symbols $u \otimes v$, $u \in U, v \in V$ such that:

(a) $(u+u') \otimes v = u \otimes v + u' \otimes v$

(b) $u \otimes (v+v') = u \otimes v + u \otimes v'$ $\Big\}$ $\forall u, u' \in U$, $v, v' \in V$

(c) $\forall \alpha \in \mathbb{C}$, $(\alpha u) \otimes v = \alpha(u \otimes v) = u \otimes (\alpha v)$

If $e_1 \dots e_n$ is basis for U,

$f_1 \dots f_m$ is a basis for V,

then $\left\{ e_i \otimes f_j \;\middle|\; \begin{matrix} i=1,\dots n \\ j=1,\dots m \end{matrix} \right\}$ is basis for $U \otimes V$

$\Rightarrow \dim(U \otimes V) = (\dim U)(\dim V)$

Suppose given $\rho_1 : GL(V) \to GL(W_1)$

$\rho_2 : GL(V) \to GL(W_2)$

Then we get $\rho = \rho_1 \otimes \rho_2 : GL(V) \to GL(W_1 \otimes W_2)$

<u>In terms of group actions:</u>

$$g \cdot \sum_i u_i \otimes v_i := \sum_i (g \cdot u_i) \otimes (g \cdot v_i)$$

$g \in GL(V)$, $u_i \in W_1$, $v_i \in W_2$

In terms of matrices: $(n = \dim W_1)$

$$(\rho_1 \otimes \rho_2)(g) = \left(\begin{array}{c|c|c} \rho_1(g)_{1,1}\,\rho_2(g) & \cdots & \rho_1(g)_{1,n}\,\rho_2(g) \\ \hline \vdots & \ddots & \vdots \\ \hline \rho_1(g)_{n,1}\,\rho_2(g) & \cdots & \rho_1(g)_{n,n}\,\rho_2(g) \end{array}\right)$$

$\Rightarrow (\mathrm{char}\, \rho_1 \otimes \rho_2)(x_1, \ldots, x_r)$ $[r = \dim V]$

$= (\mathrm{char}\, \rho_1)(x_1, \ldots x_r) \cdot (\mathrm{char}\, \rho_2)(x_1, \ldots x_r)$

If W_i both rational, so is $W_1 \otimes W_2$

If W_i both polynomial, so is $W_1 \otimes W_2$

③ Duals

Given representation $\rho : GL(V) \to GL(W)$,

dual representation $\rho^\vee : GL(V) \to GL(W^*)$

$[W^* = \{ f : W \to \mathbb{C} \mid f \text{ linear} \}]$

is defined by:

$$(g \cdot f)(v) = f(g^{-1} \cdot v) \quad \forall g \in GL(V), f \in W^*, v \in V$$

comes from ρ

In terms of matrices,

transpose

$$\rho^\vee(g) = (\rho(g)^T)^{-1}$$

$$\Rightarrow (\text{char } \rho^\vee)(x_1, \dots, x_n) = (\text{char } \rho)(x_1^{-1}, \dots, x_n^{-1})$$

If W is rational, so is W^*

Generally, if W is polynomial, W^* usually is not.

<u>Def</u> · Given 2 reps W, W' of $GL(V)$, a <u>homomorphism</u> is a linear map $f: W \to W'$ s.t. $\forall g \in GL(V)$, $\forall w \in W$, $f(g \cdot w) = g \cdot f(w)$

In terms of ρ, ρ' it means

$$f \circ \rho(g) = \rho'(g) \circ f \quad \forall g \in GL(V).$$

also called $GL(V)$-equivariant linear map

· A homomorphism is an <u>isomorphism</u> if it is invertible. Notation: $W \cong W'$ if $\exists$ isom

[Heuristically, isomorphic reps are "the same" but often W, W' might look very different]

<u>Def</u>. A <u>subrepresentation</u> of W is a subspace $U \subseteq W$ s.t. $g \cdot u \in U$ $\forall g \in GL(V)$ $\forall u \in U$

If $W \neq 0$, then W is called <u>irreducible</u> (<u>simple</u>) if its only subrepresentations are 0 & W.

W is called <u>semisimple</u> if it is isomorphic to a direct sum of simple representations.

<u>Thm</u> ① Every finite-dim representation of $GL(V)$ is semisimple.

② Two (finite-dim) representations of $GL(V)$ are isomorphic iff they have equal characters.

③ (Schur's lemma) Let W, W' be 2 simple representations which are not isomorphic. Then every homomorphism $f: W \to W'$ must be 0.

Every homomorphism $f: W \to W$ is a scalar multiple of the identity function.

Given 2 representations W, W' we can write both as direct sums of simples, then Schur's lemma constrains possible <u>homomorphisms</u> $f: W \to W'$.

Furthermore, the simples appearing in a direct sum decomposition are uniquely determined: if $W \cong \bigoplus_{i \in I} W_i^{\oplus m_i} \cong \bigoplus_{j \in J} U_j^{\oplus n_j}$

where W_i, U_j are simple, then there is a bijection $\sigma: I \xrightarrow{\cong} J$ s.t.

$W_i \cong U_{\sigma(i)}$ & $m_i = n_{\sigma(i)}$. $\forall i$.

Highest weight vectors

Pick a Borel subgroup $B \subset GL(V)$ and a maximal torus $T \subset B$.

Def. A nonzero vector $w \in W$ is a highest weight vector if $b \cdot w$ is a scalar multiple of w $\forall b \in B$, ie, w is an eigenvector for $\rho(b)$ $\forall b \in B$.

$\Longrightarrow$ since $T \subset B$, highest weight vectors are weight vectors.

If μ is the weight of w, then write $\mu(b)$ for the scalar s.t. $b \cdot w = \mu(b) \cdot w$.

Examples ① $V = W = \mathbb{C}^n$, $\rho = id$.

$B =$ upper-triangular matrices.

$$= \begin{pmatrix} * & * & * & & * \\ & * & * & & * \\ & & * & \ddots & * \\ & 0 & & & * \end{pmatrix}$$

e_1 is a highest weight vector, and any other must be multiple of e_1

Weight is $(1, 0, \dots, 0)$

② $V = W = \mathbb{C}^n$, $\rho = \mathrm{id}^\vee$

$\rho(b)$ is then lower-triangular $\forall b \in B$

$$\begin{pmatrix} * & & & 0 \\ * & * & & \\ \vdots & * & \ddots & \\ * & \vdots & \cdots & * \end{pmatrix}$$

e_n^* is a highest weight vector, and any other is a multiple of e_n^*

weight is $(0, 0, \ldots, 0, -1)$.

③ $V = \mathbb{C}^2$, $W = \mathbb{C}^3$

e_1 is a highest weight vector and again, unique up to scalar multiple.

weight $= (2, 0)$.

④ $\mathbb{C}^n = V$, $W = \mathbb{C}$, $\rho(g) = (\det g)^d$.

$1 \in W$ is highest weight vector,

weight is $(d, d, \ldots, d)$.