Math 251C, Lecture 18

Note Title 5/8/2020

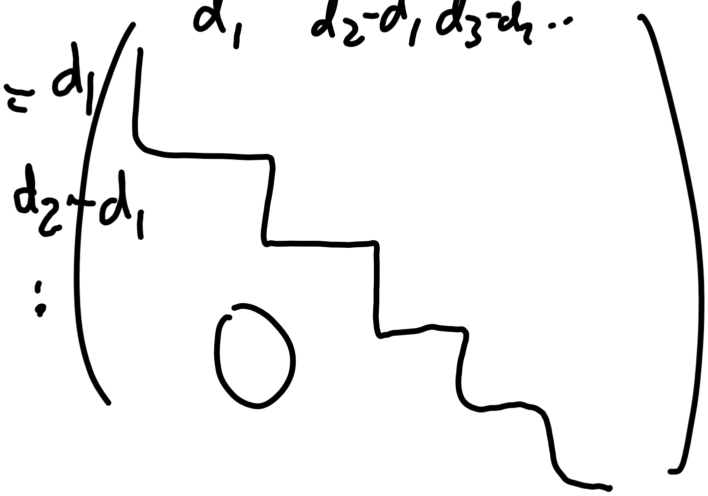
Prop. $\dim Fl(\underline{d}; n) = \sum_{i=1}^{r} (d_i - d_{i-1})(n - d_i)$

and $Fl(\underline{d}; n)$ is irreducible.

In particular, $\dim Fl(n) = \binom{n}{2}$.

Pf Consider flag where $W_{d_i} = \mathrm{span}(e_1, \dots, e_{d_i})$.

$\mathrm{Stab}(W_{d_1} \subset \cdots \subset W_{d_r}) =$



$\Rightarrow \dim Fl(\underline{d}, n) = \#\, 0\text{'s}$

$$= \sum_{i=1}^{r} (d_i - d_{i-1})(n - d_i) \qquad \square$$

Another fact: given an algebraic map $\pi : X \to Y$ between irred. varieties, s.t. $\dim \pi^{-1}(y) = e \;\forall y$,

$\Rightarrow \dim X = e + \dim Y$.

Note, if $\underline{d}'$ is a subsequence of $\underline{d}$, have forgetful map $Fl(\underline{d};n) \longrightarrow Fl(\underline{d}';n)$.

Fibers are identified w/ products of flag varieties, can use to compute $\dim Fl(\underline{d};n)$ by induction.

For $Fl(n)$, stabilizer of any pt is a Borel subgroup. Also, every Borel subgroup is stabilizer of some pt.

$$\left\{ \begin{array}{c} \text{Borel subgroups} \\ \text{of } GL_n\mathbb{C} \end{array} \right\} \longleftrightarrow Fl(n)$$

GL_n/B

variety of Borel subgroups.

Any closed proper subgroup of $GL_n\mathbb{C}$ that contains a Borel subgroup is called <u>parabolic</u>

stabilizers of partial flags are parabolic, <u>and</u> they exhaust all of them.