Math 251C, Lecture 16

Note Title 5/4/2020

<u>Cor.</u> (Orthogonal Cauchy identity) We have an isomorphism of $GL_n\mathbb{C} \times SO_m\mathbb{C}$-reps:

$$\mathbb{C}[X] \cong \bigoplus_{\lambda} S_\lambda \mathbb{C}^m \otimes S_{[\lambda]} \mathbb{C}^m$$

$\lambda_1 \geq \cdots \geq \lambda_n \geq 0$

<u>Note</u>: if $m=2n$, $\lambda_n>0$, $S_{[\lambda]}\mathbb{C}^m = S_{[\lambda]^+}\mathbb{C}^m \oplus S_{[\lambda]^-}\mathbb{C}^m$

- $\dim X = \dim U - \binom{n+1}{2}$
- $X = Z(\underbrace{\text{entries of symmetric matrix } u^T I' u}_{\text{regular sequence}})$

Serre's criterion $\Longrightarrow$ $\mathbb{C}[X] = \mathbb{C}[U]/(\text{entries of } u^T I' u)$

- Weyl's construction: $S_{[\lambda]}(\mathbb{C}^m)$ is quotient of $S_\lambda\mathbb{C}^m$ by images of sum of $S_\mu \mathbb{C}^m$ where λ/μ is 2 boxes, not in same column.
- $\mathbb{C}[X] \otimes \mathrm{Sym}(S^2\mathbb{C}^n) \cong \mathbb{C}[U]$

Thm (Stable branching rule from GL to O)

λ partition w/ $\ell(\lambda) \leq n$. Isomorphism of $O_m\mathbb{C}$-reps:

$$S_\lambda(\mathbb{C}^m) \cong \bigoplus_\mu \left(S_{[\mu]}\mathbb{C}^m\right)^{\oplus m_{\lambda\mu}} \quad \text{where}$$

$$m_{\lambda\mu} = \sum_\nu c^\lambda_{\mu, 2\nu}.$$

Ex. $\lambda = (d)$. $c^\lambda_{\mu, 2\nu} > 0 \Rightarrow \mu = (e),\ 2\nu = (2f)$
$e + 2f = d$

$$\mathrm{Sym}^d \mathbb{C}^m \cong S_{[d]}\mathbb{C}^m \oplus S_{[d-2]}\mathbb{C}^m \oplus S_{[d-4]}\mathbb{C}^m \oplus \cdots$$

$\lambda = (1^d)$ $c^\lambda_{\mu, 2\nu} > 0 \Rightarrow 2\nu = \emptyset,\ \mu = (1^d)$

$\bigwedge^d \mathbb{C}^m \cong S_{[1^d]}\mathbb{C}^m$ for $d \leq n$.

Characters of the special orthogonal group

$\rho: SO_m\mathbb{C} \to GL(W)$ rep.

$$(\mathrm{char}\ \rho)(x_1, \dots x_n) = \mathrm{trace}\ \rho\begin{pmatrix} x_1 & & & & & & 0 \\ & \ddots & & & & & \\ & & x_n & & & & \\ & & & (1?) & & & \\ & & & & x_n^{-1} & & \\ & & & & & \ddots & \\ 0 & & & & & & x_1^{-1} \end{pmatrix}$$

H_n = hyperoctahedral group.
= permutations of $\{x_1, \dots, x_n, x_n^{-1}, \dots x_1^{-1}\}$
s.t. $\sigma(1/x_i) = 1/\sigma(x_i)$

H_n' = subgroup of H_n s.t. #i s.t. $\sigma(x_i) = 1/x_j$
for some j is even.

$[H_n : H_n'] = 2$, $\quad \# H_n' = 2^{n-1} n!$

Lemma. If $m = 2n+1$, $\mathrm{char}(\rho)$ is invariant under H_n. If $m = 2n$, $\mathrm{char}\,\rho$ is invariant under H_n'.

Pf. H_n gen. by $\mathfrak{S}_n$ and $t_1 \dots t_n$, where

$$t_i(x_i) = 1/x_i, \quad t_i(x_j) = x_j \text{ for } j \neq i.$$

For $\sigma \in \mathfrak{S}_n$, action of σ realized by conjugation by $\begin{pmatrix} \sigma & 0 & 0 \\ 0 & (1?) & 0 \\ 0 & 0 & I'\sigma I' \end{pmatrix} \in SO_m\mathbb{C}$. For t_i, define an element in $O_m\mathbb{C}$ via $e_j \to e_j$ for $j \neq \pm i$

$$e_i \to e_{-i}$$
$$e_{-i} \to e_i$$

Conjugation by this element is same as applying t_i, but it has $\det = -1$.

If $m = 2n+1$, can modify by $e_{n+1} \to -e_{n+1}$

conjugation has same effect, belongs to $SO_{2n+1}\mathbb{C}$.

If $m = 2n$, product of even # of these elements does belong to $SO_{2n}\mathbb{C}$. $\square$

Cor If m is odd, or divisible by 4, then every rep of $SO_m\mathbb{C}$ is self-dual.

Pf. If $f = \operatorname{char} V$, then $f(x_1^{-1}, \ldots, x_n^{-1}) = \operatorname{char} V^*$.

In odd case, can invert all variables w/ element of H_n. If $m = 2n$, n even, can also invert all variables w/ element of H_n'. $\square$

What if $m \equiv 2 \bmod 4$? Same argument shows that reps of $O_m\mathbb{C}$ are self-dual. But, if $\ell(\lambda) = n$,

$(S_{[\lambda]+}\mathbb{C}^m)^* \cong S_{[\lambda]-}\mathbb{C}^m$.

Thm (Weyl character formula for $SO_{2n+1}(\mathbb{C})$).
$\rho = (n-\frac{1}{2}, n-\frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2})$ Then

$$s_{[\lambda]}(x_1, \ldots x_n) = \frac{\det\left(x_i^{\lambda_j+n-j+\frac{1}{2}} - x_i^{-(\lambda_j+n-j+\frac{1}{2})}\right)_{i,j=1}^{n}}{\prod_{i=1}^{n}(x_i^{1/2}-x_i^{-1/2})\prod_{1\le i<j\le n}(x_i+x_i^{-1}-x_j-x_j^{-1})}$$

$$= \frac{\sum_{\sigma\in H_n} \operatorname{sgn}(\sigma)\,\sigma(x^{\lambda+\rho})}{\sum_{\sigma\in H_n} \operatorname{sgn}(\sigma)\,\sigma(x^{\rho})}.$$

$\rightarrow$ Orthogonal Cauchy identity in character rings for $\Lambda(n) \otimes \Lambda_{O(m)}$

we define $\Lambda_{O(m)} = \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]^{H_n}$

for m even and odd.

As before, we have

$$s_\lambda(x_1, \ldots x_n, (1?), x_n^{-1}, \ldots, x_1^{-1}) = s_{[\lambda]}(x_1, \ldots, x_n) + \sum_{\substack{\mu \\ |\mu|<|\lambda|}} m_{\lambda\mu} s_{[\mu]}(x_1, \ldots, x_n)$$

$$\Longrightarrow \quad s_{[\lambda]}(x_1,\dots x_n) = \sum n_{\lambda\mu}\, s_\lambda(x_1,\dots x_n,(1?),x_n^{-1},\dots,x_1^{-1})$$

where $(n_{\lambda\mu}) = (m_{\lambda\mu})^{-1}$.

$\Longrightarrow$ orthogonal Schur functions

$$s_{[\lambda]} = \sum n_{\lambda\mu}\, s_\lambda \in \Lambda$$

note: these are not the same as symplectic Schur functions!

Ex. $s_\phi = s_{[\phi]}$

$s_1 = s_{[1]}$

$s_2 = s_{[2]} + s_{[\phi]}$ $\quad\Longrightarrow\quad s_{[2]} = s_2 - s_\phi$

$s_3 = s_{[3]} + s_{[1]}$

$s_{21} = s_{[21]} + s_{[1]}$ $\quad\Longrightarrow\quad s_{[3]} = s_3 - s_1$, $\quad s_{[21]} = s_{21} - s_1$

$s_{111} = s_{[111]}$

$$\pi_{O(m)} : \Lambda \longrightarrow \Lambda_{O(m)}$$

$$s_\lambda \longmapsto s_\lambda(x_1,\dots,x_n,(1?),x_n^{-1},\dots,x_1^{-1})$$

$$\pi_{O(m)}(s_{[\lambda]}) = s_{[\lambda]}(x_1,\dots x_n)$$

$\exists\ {}^{O}c^{\lambda}_{\mu\nu} \in \mathbb{Z}$ s.t. $s_{[\mu]} s_{[\nu]} = \sum_{\lambda} {}^{O}c^{\lambda}_{\mu\nu}\, s_{[\lambda]}$

Thm (Newell–Littlewood product formula)

$${}^{O}c^{\lambda}_{\mu\nu} = \sum_{\alpha,\beta,\gamma} c^{\mu}_{\alpha\beta}\, c^{\nu}_{\alpha\gamma}\, c^{\lambda}_{\beta\gamma}.$$

Note. ${}^{Sp}c^{\lambda}_{\mu\nu} = {}^{O}c^{\lambda}_{\mu\nu}$!

Ex. $\mu = \nu = (1)$

$$\mathbb{C}^m \otimes \mathbb{C}^m \cong \Lambda^2 \mathbb{C}^m \oplus S^2 \mathbb{C}^m \cong S_{[11]}\mathbb{C}^m \oplus S_{[2]}\mathbb{C}^m \oplus \mathbb{C}$$

$${}^{O}c^{11}_{1,1} = {}^{O}c^{2}_{1,1} = {}^{O}c^{1}_{1,1} = 1$$

Symplectic case

$\mu = \nu = (1)$

$$\mathbb{C}^{2n} \otimes \mathbb{C}^{2n} \cong \Lambda^2 \mathbb{C}^{2n} \oplus S^2 \mathbb{C}^{2n} \cong S_{[1,1]}\mathbb{C}^{2n} \oplus \mathbb{C} \oplus S_{[2]}\mathbb{C}^{2n}$$

$${}^{Sp}c^{11}_{1,1} = {}^{Sp}c^{2}_{1,1} = {}^{Sp}c^{1}_{1,1} = 1.$$