Math 251C, Lecture 14

Note Title 4/29/2020

$$\pi_{Sp(2n)} : \Lambda \longrightarrow \Lambda_{Sp(2n)}$$

$$s_\lambda \longmapsto s_\lambda(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}).$$

If $n \geq \ell(\lambda)$, then $\pi_{Sp(2n)}(s_{[\lambda]}) = s_{[\lambda]}(x_1, \dots, x_n)$

Two identities:

__Lemma__ (GL-branching rule) $GL_{n+m} \downarrow_{GL_n \times GL_m}$

$$s_\lambda(x_1, \dots, x_n, y_1, \dots y_m) = \sum_{\mu, \nu} c^\lambda_{\mu\nu} s_\mu(x_1 \dots x_n) s_\nu(y_1 \dots y_m)$$

__Pf.__ Pick $r >> 0$.

$$\sum_\lambda s_\lambda(x_1 \dots x_n, y_1 \dots y_m) s_\lambda(z_1 \dots z_r) t^{|\lambda|} \overset{\text{Cauchy}}{=} \prod_{i=1}^r \left(\prod_{j=1}^n (1 - x_j z_i t) \prod_{j=1}^m (1 - y_j z_i t)\right)$$

$$= \left(\sum_\mu s_\mu(x_1 \dots x_n) s_\mu(z_1 \dots z_r) t^{|\mu|}\right) \cdot \left(\sum_\nu s_\nu(y_1 \dots y_m) s_\nu(z_1 \dots z_r) t^{|\nu|}\right)$$

$$= \sum_{\mu, \nu} s_\mu(x_1 \dots x_n) s_\nu(y_1 \dots y_m) \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(z_1, \dots, z_r) t^{|\lambda|}.$$

Compare coefficients $\square$

Lemma (Modified symplectic Cauchy identity)

$$\frac{\prod_{1\le i<j\le r}(1-y_iy_jt^2)}{\prod_{i=1}^{n}\prod_{j=1}^{r}(1-x_iy_jt)(1-x_i^{-1}y_jt)} = \sum_{\lambda} \pi_{Sp(2n)}(s_{[\lambda]})\, s_\lambda(y_1 \ldots y_r)\, t^{|\lambda|}$$

$\lambda_1 \ge \cdots \ge \lambda_r \ge 0$

Pf. Suffices to show this when r fixed, $n >> 0$.

If $n \ge 2r$, take symplectic Cauchy identity for $Sp(2n) \times GL(n)$ and set $y_{r+1} = y_{r+2} = \cdots = y_n = 0$ □

Since $\{s_{[\lambda]}\}$ basis for Λ, $\exists$ integers ${}^{Sp}c^{\lambda}_{\mu\nu}$ s.t.

$$s_{[\mu]}s_{[\nu]} = \sum_{\lambda} {}^{Sp}c^{\lambda}_{\mu\nu}\, s_{[\lambda]}$$

Thm (Newell-Littlewood product formula)

$$\forall n,\ s_{[\mu]}(x_1 \ldots x_n)\, s_{[\nu]}(x_1 \ldots x_n) = \sum_{\lambda,\alpha,\beta,\gamma} c^{\lambda}_{\beta\gamma}\, c^{\mu}_{\beta\alpha}\, c^{\nu}_{\alpha\gamma}\, \pi_{Sp(2n)}(s_{[\lambda]})$$

In particular,

$$ {}^{Sp}c^{\lambda}_{\mu\nu} = \sum_{\alpha,\beta,\gamma} \underbrace{c^{\lambda}_{\beta\gamma}\, c^{\mu}_{\beta\alpha}\, c^{\nu}_{\alpha\gamma}}_{\text{sum over all partitions}}.$$

Pf. $$\frac{\displaystyle\prod_{1\le i<j\le 2n}(1-y_iy_jt^2)}{\displaystyle\prod_{i=1}^{n}\prod_{j=1}^{2n}(1-x_iy_jt)(1-x_i^{-1}y_jt)}\cdot\prod_{i=1}^{n}\prod_{j=n+1}^{2n}\frac{1}{(1-y_iy_jt^2)}$$

$$=\Bigg(\sum_{\substack{\lambda\\ \ell(\lambda)\le 2n}}\pi_{Sp(2n)}(s_{[\lambda]})\,\underbrace{s_\lambda(y_1,\dots y_{2n})}_{\text{GL-branching}}\,t^{|\lambda|}\Bigg)\cdot$$

Modified symplectic Cauchy

$$\Bigg(\sum_{\alpha}s_\alpha(y_1\dots y_n)\,s_\alpha(y_{n+1},\dots y_{2n})\,t^{2|\alpha|}\Bigg).$$

Cauchy

$$=\Bigg(\sum_{\substack{\lambda,\beta,\gamma\\ \ell(\lambda)\le 2n}}\pi_{Sp(2n)}(s_{[\lambda]})\,c^{\lambda}_{\beta\gamma}\,\underline{s_\beta(y_1,\dots y_n)}\,\underline{s_\gamma(y_{n+1},\dots y_{2n})}\,t^{|\lambda|}\Bigg)$$

$$\cdot\Bigg(\sum_{\alpha}\underline{s_\alpha(y_1\dots y_n)}\,\underline{s_\alpha(y_{n+1},\dots y_{2n})}\,t^{2|\alpha|}\Bigg).$$

$$=\sum_{\substack{\lambda,\beta,\gamma\\ \mu,\nu}}\Bigg(\pi_{Sp(2n)}(s_{[\lambda]})\,c^{\lambda}_{\beta\gamma}\,c^{\mu}_{\alpha\beta}\,s_\mu(y_1,\dots y_n)\cdot c^{\nu}_{\alpha\gamma}\,s_\nu(y_{n+1},\dots y_{2n})\,t^{|\mu|+|\nu|}\Bigg)$$

First expression factors as:

$$\frac{\prod_{1\le i<j\le n}(1-y_iy_jt^2)}{\prod_{i=1}^{n}\prod_{j=1}^{n}(1-x_iy_jt)(1-x_i^{-1}y_jt)}\;\frac{\prod_{n+1\le i<j\le 2n}(1-y_iy_jt^2)}{\prod_{i=1}^{n}\prod_{j=n+1}^{2n}(1-x_iy_jt)(1-x_i^{-1}y_jt)}.$$

↓ symplectic Cauchy identity

$$=\left(\sum_{\mu} s_{[\mu]}(x_1\dots x_n)\,s_\mu(y_1\dots y_n)\,t^{|\mu|}\right)\left(\sum_{\nu} s_{[\nu]}(x_1\dots x_n)\,s_\nu(y_{n+1}\dots y_{2n})\,t^{|\nu|}\right).$$

Compare coefficients w/ last expression to get

$$\sum_{\lambda,\alpha,\beta,\gamma}\pi_{Sp(2n)}(S_{[\lambda]})\,c^{\lambda}_{\beta\gamma}\,c^{\mu}_{\alpha\beta}\,c^{\nu}_{\alpha\gamma}=s_{[\mu]}(x_1\dots x_n)\,s_{[\nu]}(x_1,\dots,x_n).\qquad\square$$

Orthogonal Groups

Recall: $GL_m\mathbb{C}$ acts on $\mathrm{Sym}^2\mathbb{C}^m$ (space of symmetric matrices) via

$$g \cdot u = (g^{-1})^T u g.$$

The stabilizer of a full rank matrix is an orthogonal group. (All orthogonal groups conjugate $\Rightarrow$ isomorphic)

$I' = \begin{pmatrix} 0 & & 1 \\ & \iddots & \\ 1 & & 0 \end{pmatrix}$, let $O_m\mathbb{C} = \mathrm{stab}(I')$.

$$O_m\mathbb{C} = \{g \mid g^T I' = I'\}$$
$$= \{g \mid I'(g^T)^{-1} I' = g\}$$

Difference w/ Sp: If $g \in Sp_{2n}\mathbb{C}$, then $\det g = 1$

If $g \in O_m\mathbb{C}$, then $\det g = \pm 1$.

Special orthogonal group $SO_m\mathbb{C}$ is $\{g \in O_m\mathbb{C} \mid \det g = 1\}$.

$$\dim O_m\mathbb{C} = \dim GL_m\mathbb{C} - \dim \mathrm{Sym}^2\mathbb{C}^m$$
$$= m^2 - \binom{m+1}{2} = \binom{m}{2}$$

I' defines bilinear form on $\mathbb{C}^m$:

$$\omega(u,v) = u^T I' v$$

$I'^T = I \Rightarrow \omega$ symmetric $\omega(u,v) = \omega(v,u)$.

I' full rank $\Rightarrow \omega$ non-degenerate, i.e., $\forall u, \exists v$ s.t. $\omega(u,v) \neq 0$.

$\Rightarrow \omega$ is orthogonal form.

wrt $e_1, \ldots, e_m$, $\omega(e_i, e_{m+1-j}) = \delta_{ij}$.

Such a basis called hyperbolic basis.

Borel = upper triangular matrices wrt. hyperbolic basis

max torus = diagonal matrices $\Big\}$ in $SO_m\mathbb{C}$

even case, torus:

$$\begin{pmatrix} x_1 & & & & & \\ & \ddots & & & 0 & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & 0 & & & \ddots & \\ & & & & & x_1^{-1} \end{pmatrix}$$

$m = 2n$

odd case

$$\begin{pmatrix} x_1 & & & & & & \\ & \ddots & & & & 0 & \\ & & x_n & & & & \\ & & & 1 & & & \\ & & 0 & & x_n^{-1} & & \\ & & & & & \ddots & \\ & & & & & & x_1^{-1} \end{pmatrix}$$

$m = 2n+1$