Math 251C, Lecture 13

Note Title 4/27/2020

Thm (Stable branching rule $GL \to Sp$) Let λ be a partition w/ $\ell(\lambda) \le n$. Then we have Sp_{2n}-rep isom:

$$S_\lambda \mathbb{C}^{2n} \cong \bigoplus_\mu \left(S_{[\mu]} \mathbb{C}^{2n}\right)^{\oplus m_{\lambda,\mu}}$$

where $m_{\lambda,\mu} = \sum_\nu c^\lambda_{\mu,(2\nu)^T}$.

Ex1. $\lambda = \mu$, $m_{\lambda,\lambda} = 1$ since only $c^\lambda_{\lambda,\emptyset} = 1$ contributes to sum

$\Rightarrow S_{[\lambda]}\mathbb{C}^{2n}$ appears w/ mult. 1 in $S_\lambda \mathbb{C}^{2n}$

In fact, h.w. vector for $GL_{2n}\mathbb{C}$ in $S_\lambda \mathbb{C}^{2n}$ is a h.w. vector for $Sp_{2n}\mathbb{C}$ of weight λ

Ex2. $\lambda = (1^d)$. $c^\lambda_{\mu,(2\nu)^T} > 0$ implies that $\mu = (1^e)$ $(2\nu)^T = (1^f)$, f even. & $e+f=d$

In that case, $c^{(1^d)}_{(1^e),(1^f)} = 1$ by Pieri rule.

$$\Rightarrow S_{(1^d)} \mathbb{C}^{2n} = \Lambda^d \mathbb{C}^{2n} \cong S_{[1^d]} \mathbb{C}^{2n} \oplus S_{[1^{d-2}]} \mathbb{C}^{2n} \oplus S_{[1^{d-4}]} \mathbb{C}^{2n} \oplus \cdots$$

<u>Ex 3</u>. unstable example:

$\lambda = (1^{2n})$ $\quad S_\lambda \mathbb{C}^{2n} = \Lambda^{2n} \mathbb{C}^{2n} = \det$

$g \in Sp_{2n}\mathbb{C} \Rightarrow \det g = 1$. hence $\Lambda^{2n} \mathbb{C}^{2n} = \text{trivial} = S_{[0]} \mathbb{C}^{2n}$

Characters of the symplectic group

$\rho : Sp_{2n}\mathbb{C} \longrightarrow GL(W)$ rep.

$$\Rightarrow (\text{char } \rho)(x_1, \ldots, x_n) = \text{trace } \rho \begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & x_1^{-1} \end{pmatrix}$$

Can see that char ρ is symmetric in x_i's.

but more is true.

Let H_n = permutations σ of $\{x_1, \ldots x_n, x_n^{-1}, \ldots, x_1^{-1}\}$

s.t. $\sigma\left(\frac{1}{x_i}\right) = \frac{1}{\sigma(x_i)}$. $\quad \mathfrak{S}_n \subset H_n$ as permutations of $\{x_1, \ldots x_n\}$

$|H_n| = 2^n n!$ $\quad$ <u>hyperoctahedral group</u>

Abstractly, $H_n \cong \mathfrak{S}_n \ltimes (\mathbb{Z}/2)^n$

<u>Lemma</u> $(\mathrm{char}\,\rho)(x_1, \dots x_n)$ is invariant under H_n.

<u>Pf</u>. H_n is generated by $\widetilde{\mathfrak{S}}_n$ and $t_1, \dots t_n$ where $t_i(x_j) = \begin{cases} x_i^{-1} & \text{if } i=j \\ x_j & \text{if } i \neq j \end{cases}$, so suffices to show invariance under these

$\sigma \in \widetilde{\mathfrak{S}}_n$: let $M(\sigma)$ be $n \times n$ permutation matrix $e_i \to e_{\sigma(i)}$.

$$\begin{pmatrix} M(\sigma) & 0 \\ 0 & I' M(\sigma) I' \end{pmatrix} \in Sp_{2n}\mathbb{C}$$

conjugation by $\tilde{\sigma}$ sends $x_i^{\pm} \to x_{\sigma(i)}^{\pm}$.

t_i : define $g_i \in Sp_{2n}\mathbb{C}$ via: $e_j \to e_j$ for $j \neq \pm i$
$e_i \to e_{-i}$
$e_{-i} \to -e_i$

$\omega(e_i, e_{-i}) = 1$
$\quad\shortparallel$
$\omega(g_i e_i, g_i e_{-i}) = \omega(e_{-i}, -e_i) = 1$

conjugation by g_i sends $x_i \to x_i^{-1}$, $x_i^{-1} \to x_i$, $x_j \to x_j$ for $j \neq \pm i$. $\square$

<u>Cor</u>. Every $Sp_{2n}\mathbb{C}$-rep is self-dual (isom. to its dual).

<u>Pf</u> If f is char. of rep, its dual has character $f(x_1^{-1}, \dots, x_n^{-1}) \overset{H_n}{=} f(x_1, \dots, x_n)$ $\square$

Define $\Lambda_{Sp(2n)} = \mathbb{Z}[x_1^{\pm}, \ldots, x_n^{\pm}]^{H_n}$ ← H_n-invariant elements

$\cup$

(char ρ) $\forall$ reps. ρ of $Sp_{2n}\mathbb{C}$.

The irreducible characters form basis for $\Lambda_{Sp(2n)}$

let $s_{[\lambda]}(x_1 \ldots x_n) = \operatorname{char} S_{[\lambda]}(\mathbb{C}^{2n})$.

Given $\sigma \in H_n \subset \mathfrak{S}_{2n}$, let $\operatorname{sgn}(\sigma)$ be its sign as an element of $\mathfrak{S}_{2n}$.

<u>Thm</u> (Weyl character formula) $\rho = (n, n-1, \ldots, 2, 1)$

$$s_{[\lambda]}(x_1 \ldots x_n) = \frac{\det\left(x_i^{\lambda_j+n-j+1} - x_i^{-(\lambda_j+n-j+1)}\right)_{i,j=1}^{n}}{\prod_{i=1}^{n}(x_i - x_i^{-1}) \prod_{1 \le i < j \le n}(x_i + x_i^{-1} - x_j - x_j^{-1})}$$

$$= \frac{\sum_{\sigma \in H_n} \operatorname{sgn}(\sigma)\, \sigma(x^{\lambda+\rho})}{\sum_{\sigma \in H_n} (\operatorname{sgn}\sigma)\, \sigma(x^{\rho})}.$$

Thm (Symplectic Cauchy identity)

$$\frac{\prod_{1\le i<j\le n}(1-y_iy_jt^2)}{\prod_{i,j=1}^{n}(1-x_iy_jt)(1-x_i^{-1}y_jt)}=\sum_{\lambda} s_{[\lambda]}(x_1\dots x_n)\,s_\lambda(y_1\dots y_n)\,t^{|\lambda|}$$

$\lambda_1\geq\dots\geq\lambda_n\geq 0\in\mathbb{Z}^n$

Pf. From mult. free action, we have

$$\mathbb{C}[X]\cong\bigoplus_{\lambda} S_{[\lambda]}\mathbb{C}^{2n}\otimes S_\lambda\mathbb{C}^n$$

$$\downarrow \text{char}$$

$$\text{char}(\mathbb{C}[X])=\sum_{\lambda} s_{[\lambda]}(x_1\dots x_n)\,s_\lambda(y_1\dots y_n)\,t^{|\lambda|}$$

Also, we have

$$\mathbb{C}[X]\otimes \text{Sym}(\textstyle\bigwedge^2\mathbb{C}^n)\cong\mathbb{C}[U]$$

space of $2n\times n$ matrices

$$\downarrow$$

$$\text{char}\,\mathbb{C}[X]\cdot\prod_{1\le i<j\le n}(1-y_iy_jt^2)^{-1}=\prod_{i,j=1}^{n}(1-x_iy_jt)^{-1}\cdot(1-x_i^{-1}y_jt)^{-1}$$

Combine both. $\square$

From stable branching rule, we have formula like:

$$s_\lambda(x_1,\dots,x_n,x_n^{-1},\dots,x_1^{-1}) = s_{[\lambda]}(x_1\dots x_n) + \sum_{\substack{\mu \\ |\mu|<|\lambda|}} m_{\lambda,\mu}\, s_{[\mu]}(x_1,\dots,x_n)$$

Order partitions of size $\leq |\lambda|$ in a way that refines size. $\Rightarrow$ matrix M whose elements are $m_{\lambda\mu}$ it is uppertriangular w/ 1's on the diagonal

$\Rightarrow$ invertible. Let $N=(n_{\lambda\mu})=M^{-1}$

$$\Rightarrow s_{[\lambda]}(x_1\dots x_n) = \sum_\mu n_{\lambda\mu}\, s_\mu(x_1\dots x_n, x_n^{-1},\dots,x_1^{-1})$$

$(n_{\lambda\lambda}=1,\ n_{\lambda\mu}\neq 0 \Rightarrow |\mu|\leq|\lambda|)$

Also: $m_{\lambda\mu}$ is independent of n if $n \geq \ell(\lambda)$.

$\Rightarrow$ $n_{\lambda\mu}$ is independent of n if $n\geq \ell(\lambda)$.

IDEA: Define $s_{[\lambda]} \in \Lambda$ $\leftarrow$ symmetric functions

via $s_{[\lambda]} = \sum_\mu n_{\lambda\mu}\, s_\mu$

symplectic Schur functions

Schur function

By upper-triangularity $\{s_{[\lambda]}\}$ is basis for Λ.

Can define specialization maps

$$\pi_{Sp(2n)} : \Lambda \longrightarrow \Lambda_{Sp(2n)}$$
$$s_\lambda \longmapsto s_\lambda(x_1, \dots, x_n, x_n^{-1}, \dots, x_1^{-1}).$$

If $n \geq \ell(\lambda)$, then $\pi_{Sp(2n)}(s_{[\lambda]}) = s_{[\lambda]}(x_1, \dots, x_n)$

But note: $\pi_{Sp(2n)}(s_{[\lambda]})$ makes sense even if $n < \ell(\lambda)$

<u>Next time</u>: study tensor products of symplectic reps