Math 251C, Lecture 12

Note Title 4/24/2020

$X = \{ u \in \mathcal{U} \mid u^T \Omega u = 0 \}$. $\mathcal{U} = 2n \times n$ matrices

solution of $\binom{n}{2}$ quadratic polynomials

$\dim X = \dim \mathcal{U} - \binom{n}{2}$

Krull's principal ideal Thm: $X \subset Y$ affine varieties

s.t. $X = Z(f_1, \dots f_r)$ $f_i \in \mathbb{C}[Y]$

$\Rightarrow \dim X \geq \dim Y - r$

If equality holds, $f_1, \dots, f_r$ is called <u>system of parameters</u>. If furthermore, Y is vector space, s.o.p. $\Rightarrow$ <u>regular sequence</u>. (def omitted)

<u>Thm</u> U = vector space, $f_1, \dots f_r \in \mathbb{C}[U]$ regular sequence of homogeneous polynomials.

Pick basis $\{g_i'\}$ of $\mathbb{C}[U]/(f_1, \dots f_r)$, and pick coset representatives $g_i \in \mathbb{C}[U]$. Then

$$\mathbb{C}[U]/(f_1, \dots f_r) \otimes \mathbb{C}[y_1, \dots, y_r] \longrightarrow \mathbb{C}[U]$$
$$g_i' \otimes p(y) \longrightarrow g_i \, p(f_1, \dots f_r)$$

is an isomorphism of vector spaces.

Cor. Let U be representation. $f_1,\dots f_r \in \mathbb{C}[U]$ regular sequence of homogeneous polynomials s.t. they span a subrepresentation $V \subseteq \mathbb{C}[U]$. Then we have an isomorphism of representations

$$\mathbb{C}[U]/(f_1,\dots,f_r) \otimes \mathrm{Sym}(V) \cong \mathbb{C}[U].$$

Our situation:

U = $2n \times n$ matrices w/ action of $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$.

$r = \binom{n}{2}$, $f_1,\dots,f_r$ = upper-triangular entries of $u^T\Omega u$

$V \cong \Lambda^2 \mathbb{C}^n$ where $Sp_{2n}\mathbb{C}$ acts trivially.

$\Rightarrow$ get isom. of $GL_n\mathbb{C} \times Sp_{2n}\mathbb{C}$-reps

$$\mathbb{C}[U]/(f_1,\dots,f_r) \otimes \mathrm{Sym}(\Lambda^2 \mathbb{C}^n) \cong \mathbb{C}[U]$$

want to identify w/ $\mathbb{C}[X]$

Use mult. free action to write in terms of Schur functors

Cauchy identity to write in terms of Schur functors

$X = Z(f_1, \dots f_r)$ However, $\mathbb{C}[X] = \mathbb{C}[U]/I$

where I = ideal of <u>all</u> polynomials f s.t.

$f(u) = 0 \ \forall u \in X$ ($f_1, \dots f_r \in I$, but do they generate?)

<u>Example</u>. U = 2x2 matrices.

$X = \{u \in U \mid u^2 = 0\}$

entries of u^2 give 4 quadratic polynomials $f_1 \dots f_4$

s.t. $X = Z(f_1, \dots f_4) \Rightarrow X$ affine variety

Is $\mathbb{C}[X] = \mathbb{C}[U]/(f_1, f_2, f_3, f_4)$?

i.e., if $f(u) = 0 \ \forall u \in X \overset{?}{\Longrightarrow} f \in (f_1, \dots f_4)$?

<u>No</u>: all polynomials in $(f_1, \dots f_4)$ have $\deg \geq 2$.

But, $\operatorname{trace}(u) = 0$ if $u^2 = 0$. & $\deg(\operatorname{trace}) = 1$.

So $\operatorname{trace} \notin (f_1, \dots f_4)$.

In fact, $\mathbb{C}[X] = \mathbb{C}[U]/(\operatorname{trace}, \det)$

Prop. Any polynomial that is identically 0 on X is generated by entries of $u^T \Omega u$. In particular, $\mathbb{C}[X] = \mathbb{C}[u]/(f_1, \dots f_{\binom{n}{2}})$.

Can be proven using:

Thm (Serre's criterion, special case)
U = vector space, $X \subset U$ affine variety s.t.
$X = Z(f_1, \dots f_r)$ where $f_1, \dots f_r \in \mathbb{C}[U]$ regular sequence.
Let $x_1, \dots x_N$ basis for U^*. Consider
Jacobian matrix $\left(\frac{\partial f_i}{\partial x_j}\right)_{i=1, j=1}^{r, N}$.

Let $X_1, \dots, X_d$ be irreducible components of X
($X = X_1 \cup \dots \cup X_d$, X_i = maximal w.r.t. being irreducible)

If $\exists \alpha_k \in X_k$ s.t. $\left(\frac{\partial f_i}{\partial x_j}(\alpha_k)\right)$ has maximal rank r for all $k = 1, \dots, d$, then $\mathbb{C}[X] = \mathbb{C}[U]/(f_1 \dots f_r)$.

For our situation X irreducible so $d = 1$.
Take $\alpha = J$.

Weyl's construction for $S_{[\lambda]}\mathbb{C}^{2n}$

$\mathbb{C}[X]$ is a quotient of $\mathbb{C}[u]$. kernel is generated by $\Lambda^2\mathbb{C}^n$, in particular kernel is a quotient of $\mathbb{C}[u]\otimes\Lambda^2\mathbb{C}^n \rightsquigarrow$

$$\mathbb{C}[u]\otimes\Lambda^2\mathbb{C}^n \longrightarrow \mathbb{C}[u] \longrightarrow \mathbb{C}[X] \longrightarrow 0$$

Take the $S_\lambda\mathbb{C}^n$-component of these spaces as $Gl_n\mathbb{C}$-rep.

$$\bigoplus_\mu (S_\mu\mathbb{C}^{2n})^{\oplus c^\lambda_{\mu,(1,1)}} \longrightarrow S_\lambda\mathbb{C}^{2n} \longrightarrow S_{[\lambda]}\mathbb{C}^{2n} \longrightarrow 0$$

Pieri rule: this is 1 if $\lambda \supseteq \mu$ and λ/μ is 2 boxes in different rows and 0 otherwise

Recipe: Given μ s.t. we have

$$S_\mu\mathbb{C}^{2n} \longrightarrow S_\mu\mathbb{C}^{2n}\otimes\Lambda^2\mathbb{C}^{2n} \longrightarrow S_\lambda\mathbb{C}^{2n}$$

$$v \longmapsto v\otimes\Omega$$

$Sp_{2n}\mathbb{C}$-eqvt

Pieri rule, GL_{2n}-eqvt

We get $S_\mu \mathbb{C}^{2n} \longrightarrow S_\lambda \mathbb{C}^{2n}$ $Sp_{2n}\mathbb{C}$-eqvt

$\Rightarrow S_{[\lambda]}\mathbb{C}^{2n}$ is quotient of $S_\lambda \mathbb{C}^{2n}$ by submodule spanned by images as we range over all μ s.t. λ/μ is 2 boxes in different rows.

<u>So far</u>: $\mathbb{C}[X] \otimes \mathrm{Sym}(\wedge^2 \mathbb{C}^n) \cong \mathbb{C}[U]$

$$\underbrace{\left(\bigoplus_\mu S_\mu \mathbb{C}^n \otimes S_{[\mu]}\mathbb{C}^{2n}\right) \otimes \left(\bigoplus_\nu S_{(2\nu)^T}\mathbb{C}^n\right)} \cong \bigoplus_\lambda S_\lambda \mathbb{C}^n \otimes S_\lambda \mathbb{C}^{2n}$$

$$\bigoplus_{\substack{\mu,\nu\\ \lambda}} \left(S_\lambda \mathbb{C}^n \otimes S_{[\mu]}\mathbb{C}^{2n}\right)^{\oplus c^\lambda_{\mu,(2\nu)^T}}$$

<u>Thm</u> (Stable branching rule $GL \to Sp$) Let λ be a partition w/ $\ell(\lambda) \le n$. Then we have Sp_{2n}-rep isom:

$$S_\lambda \mathbb{C}^{2n} \cong \bigoplus_\mu \left(S_{[\mu]}\mathbb{C}^{2n}\right)^{\oplus m_{\lambda,\mu}}$$

where $m_{\lambda,\mu} = \sum_\nu c^\lambda_{\mu,(2\nu)^T}$.

Pf. Take $S_\lambda \mathbb{C}^n$-components of eqn above.