Math 251C, Lecture 10

Note Title 4/20/2020

$$I' = \left.\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}\right\}n \qquad \Omega = \begin{pmatrix} 0 & I' \\ -I' & 0 \end{pmatrix} \in \Lambda^2 \mathbb{C}^{2n}$$

$$\begin{aligned} Sp_{2n}\mathbb{C} &= \operatorname{stab}(\Omega) \\ &= \{g \mid g^T \Omega g = \Omega\} \\ &= \{g \mid \Omega^{-1}(g^T)^{-1}\Omega = g\} \quad (\text{since } \Omega^{-1} = -\Omega^T) \end{aligned}$$

$\Longrightarrow$ $Sp_{2n}\mathbb{C}$ is an affine variety in $4n^2$-dim space

For algebraic varieties, there is notion of (Krull) dimension. Some properties:

1. $\dim GL_n\mathbb{C} = n^2$
2. If X is vector space, $\dim X$ = dimension of X as vector space
3. If X irreducible, $U \subseteq X$ open nonempty, then $\dim U = \dim X$

④ If U is an orbit of G and H is the stabilizer of a point in U, $\dim G - \dim H = \dim U$

⑤ $\dim (X \times Y) = \dim X + \dim Y$.

$$\dim Sp_{2n}\mathbb{C} \overset{4}{=} \dim GL_{2n}\mathbb{C} - \dim \text{orbit of } \Omega$$
$$\overset{3}{=} \dim GL_{2n}\mathbb{C} - \dim \bigwedge^2 \mathbb{C}^{2n}$$
$$\overset{1,2}{=} (2n)^2 - \binom{2n}{2}$$
$$= n(2n+1)$$

Ω defines a bilinear form on $\mathbb{C}^{2n}$:

$$\omega(u,v) := u^T \Omega v$$

Since $\Omega = -\Omega^T$, $\omega(u,v) = -\omega(v,u)$

Since $\operatorname{rk}\Omega = 2n$, ω is non-degenerate, i.e.,

$\forall u \neq 0$, $\exists v$ s.t. $\omega(u,v) \neq 0$.

ω is a symplectic form

$e_1, \ldots, e_{2n}$ = standard basis for $\mathbb{C}^{2n}$:

$$\omega(e_i, e_{2n+1-j}) = \delta_{ij} \quad \text{for } 1 \leq i \leq n$$

Any basis w/ these pairings is a <u>symplectic basis</u>

A representation is homomorphism $\rho: Sp_{2n}\mathbb{C} \to GL(W)$ s.t. entries of ρ are rational functions of entries of $Sp_{2n}\mathbb{C}$.

Given a symplectic basis, we let B = subgroup in Sp_{2n} of uppertriangular matrices wrt this basis (Borel subgroup)

T = diagonal matrices (maximal torus)

$$= \left\{ \begin{pmatrix} x_1 & & & & & 0 \\ & \ddots & & & & \\ & & x_n & & & \\ & & & x_n^{-1} & & \\ & & & & \ddots & \\ 0 & & & & & x_1^{-1} \end{pmatrix} \;\middle|\; x_i \in \mathbb{C} \right\}$$

weight vectors, weights, characters defined in same way, and highest weight vectors.

Tensor product, direct sums, duals also same.

<u>Thm</u>
- Finite dim reps are semisimple,
- Reps determined by their characters
- Schur's lemma holds
- irred reps $\longleftrightarrow$ dominant weight (= weights which are highest weight of some f.d. rep)
- Dominant weights $(\mu_1, \ldots \mu_n) \in \mathbb{Z}^n$ are characterized by $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n \geq 0$

Examples ① $\rho : Sp_{2n}\mathbb{C} \hookrightarrow GL_{2n}\mathbb{C}$ inclusion

$e_1 \dots e_{2n}$ weight vectors

$$e_i \longleftrightarrow (0, \dots \underset{i}{1}, \dots 0) \qquad (1 \leq i \leq n)$$

$$e_{2n+1-i} \longleftrightarrow (0, \dots -1, \dots 0)$$

h.w. vector is e_1, hw $= (1, 0, \dots 0)$

$$\text{char}\, \rho(x_1 \dots x_n) = x_1 + \dots + x_n + x_n^{-1} + \dots + x_1^{-1}$$

Note: $(\mathbb{C}^{2n}) \cong (\mathbb{C}^{2n})^*$ b/c $\text{char}\,\rho(x_1^{-1}, \dots, x_n^{-1}) = \text{char}\,\rho$

② Given a rational rep $\rho : GL_{2n}\mathbb{C} \to GL(W)$ can restrict to $Sp_{2n}\mathbb{C}$, denote by $\rho|_{Sp_{2n}\mathbb{C}}$

$$\Longrightarrow \text{char}\,\rho|_{Sp_{2n}\mathbb{C}}(x_1 \dots x_n) = (\text{char}\,\rho)(x_1, \dots x_n, x_n^{-1}, \dots x_1^{-1})$$

③ Take $W = \Lambda^2 \mathbb{C}^{2n}$ in ② w/ $n \geq 2$.

$$(\text{char}\,\rho)(y_1 \dots y_{2n}) = \sum_{1 \leq i < j \leq 2n} y_i y_j$$

$$(\text{char}\,\rho|_{Sp_{2n}\mathbb{C}})(x_1, \dots x_n) = \sum_{1 \leq i < j \leq n} (x_i x_j + x_i^{-1} x_j^{-1}) + \sum_{i \neq j} x_i x_j^{-1} + n$$

<u>Note</u>. $\bigwedge^2 \mathbb{C}^{2n}$ is reducible for $Sp_{2n}\mathbb{C}$

line spanned by Ω is a trivial subrep!

Let $\bigwedge^2_0 \mathbb{C}^{2n}$ be a complementary subrep to Ω

<u>Claim</u>. $\bigwedge^2_0 \mathbb{C}^{2n}$ is irreducible!

only dominant weights are $(1,1,0,\ldots,0)$
and $(0,\ldots,0)$

Suffices to show no further trivial subreps in $\bigwedge^2_0 \mathbb{C}^{2n}$
(exercise)

<u>Question</u>: How does $S_\lambda \mathbb{C}^{2n} / Sp_{2n}\mathbb{C}$ decompose into irreducible reps? (branching problem)

Study using mult-free actions

Given dominant weight μ for $Sp_{2n}\mathbb{C}$, let

$S_{[\mu]}\mathbb{C}^{2n}$ = corresponding irreducible.

3: $S_{(1,1)}\mathbb{C}^{2n} \cong S_{[1,1]}\mathbb{C}^{2n} \oplus S_{[0]}\mathbb{C}^{2n}$

Prop $GL_n\mathbb{C}$ & $Sp_{2n}\mathbb{C}$ are connected in Zariski topology. (left to HW)

Def. A subspace $V \subseteq \mathbb{C}^{2n}$ is <u>isotropic</u> if $\omega(x,y)=0 \ \forall x,y \in V$.

Properties:

1. Every 1-dim space is isotropic
2. If V is isotropic, then $\dim V \leq n$

   If $\dim V = n$, called <u>Lagrangian subspace</u>

3. Given V_1, V_2 isotropic w/ same dim, $\exists g \in Sp_{2n}\mathbb{C}$ s.t. $gV_1 = V_2$
4. Every isotropic space is contained in a Lagrangian subspace

Ex. $\mathrm{span}\langle e_1, \ldots, e_n\rangle$, $\mathrm{span}\langle e_{n+1}, \ldots, e_{2n}\rangle$ are both Lagrangian subspaces