Math 251C, Spring 2020
Homework 2

(1) Show that the dual of an irreducible representation is also irreducible. Compute the highest weight of $(\mathbf{S}_\lambda \mathbf{C}^n)^*$.

Hint: find an SSYT whose basis vector is invariant under lower triangular matrices

(2) Let $h_d(x_1, \dots, x_n)$ be the character of $\mathrm{Sym}^d\,\mathbf{C}^n$.

(a) Let q be an indeterminate. Show that

$$h_d(1, q, \dots, q^n) = h_{d-1}(1, q, \dots, q^n) + q^d h_d(1, q, \dots, q^{n-1}).$$

(b) Define $[n]_q = \dfrac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$, and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \dfrac{[n]_q!}{[k]_q![n-k]_q!}$.

Show that $h_d(1, q, \dots, q^n) = \begin{bmatrix} d+n \\ d \end{bmatrix}_q$.

(c) Prove the following Hermite reciprocity identity:

$$\mathrm{Sym}^d(\mathrm{Sym}^n(\mathbf{C}^2)) \cong \mathrm{Sym}^n(\mathrm{Sym}^d(\mathbf{C}^2))$$

as $\mathbf{GL}_2(\mathbf{C})$-representations.

Hint: Reduce this to characters and do the substitution $x_1 \mapsto q$, $x_2 \mapsto 1$.

Remark: This is specific to $\mathbf{GL}_2(\mathbf{C})$ and false once we move to more variables. The Foulkes conjecture states that the multiplicity of $\mathbf{S}_\lambda(\mathbf{C}^n)$ in $\mathrm{Sym}^d(\mathrm{Sym}^e(\mathbf{C}^n))$ is at least its multiplicity in $\mathrm{Sym}^e(\mathrm{Sym}^d(\mathbf{C}^n))$ whenever $d \ge e$.

(d) Let $e_d(x_1, \dots, x_n)$ be the character of $\bigwedge^d \mathbf{C}^n$. Show that

$$e_d(1, q, \dots, q^n) = q^{\binom{d}{2}} \begin{bmatrix} n+1 \\ d \end{bmatrix}_q$$

and use this to prove $\bigwedge^d(\mathrm{Sym}^n\,\mathbf{C}^2)$ is isomorphic (up to a power of determinant) to a composition of symmetric powers.

(3) Show that the complement of the line spanned by Ω in $\bigwedge^2 \mathbf{C}^{2n}$ is an irreducible $\mathbf{Sp}_{2n}(\mathbf{C})$ representation.

(4) Prove the properties of isotropic subspaces of $\mathbf{C}^{2n}$ stated in the notes:

(a) Every 1-dimensional subspace is isotropic.

(b) If V is isotropic, then $\dim V \le n$.

(c) Given 2 isotropic subspaces V_1, V_2 with $\dim V_1 = \dim V_2$, there exists $g \in \mathbf{Sp}_{2n}(\mathbf{C})$ such that $gV_1 = V_2$.

(d) Every isotropic subspace is contained in an n-dimensional isotropic subspace.

(5) In general, irreducible implies connected, but not the converse. Show that if the space is a group (and the group operations are continuous), then connected does implies irreducible.

(6) This exercise gives generators for $\mathbf{GL}_n(\mathbf{C})$ and shows that it is connected in the Zariski topology (it easily applies also to the standard Euclidean topology).

(a) A transvection $E_{i,j;a}$ is the operator $e_k \mapsto e_k$ for $k \ne i$, and $e_i \mapsto e_i + ae_j$. Show that every $g \in \mathbf{GL}_n(\mathbf{C})$ can be written as a product of transvections and a diagonal matrix.

(b) Given $g = E_{i_1,j_1;a_1} \cdots E_{i_r,j_r;a_r} d$ with d diagonal, define $\alpha_g \colon \mathbf{C} \to \mathbf{GL}_n(\mathbf{C})$ by $\alpha_g(t) = E_{i_1,j_1;ta_1} \cdots E_{i_r,j_r;ta_r} d$. Show that α_g is continuous and conclude that every matrix is in the same connected component as a diagonal matrix.

(c) Show that the set of diagonal matrices is connected and conclude that $\mathbf{GL}_n(\mathbf{C})$ is connected.

(7) This exercise gives generators for $\mathbf{Sp}_{2n}(\mathbf{C})$ and shows that it is connected in the Zariski topology (it easily applies also to the standard Euclidean topology).

(a) Given $a \in \mathbf{C}^{2n}$ and $\lambda \in \mathbf{C}$, define

$$T_{a,\lambda}(x) = x + \lambda\omega(x,a)a.$$

Show that $T_{a,\lambda} \in \mathbf{Sp}_{2n}(\mathbf{C})$. This is called a **symplectic transvection**. Let ST be the subgroup of $\mathbf{Sp}_{2n}(\mathbf{C})$ generated by symplectic transvections.

(b) Pick $u, v \in \mathbf{C}^{2n} \setminus 0$. If $\omega(u,v) \neq 0$, we have $T_{v-u,\omega(u,v)^{-1}}(u) = v$. If $\omega(u,v) = 0$, show there exists $w \in \mathbf{C}^{2n}$ so that $\omega(u,w) \neq 0$ and $\omega(v,w) \neq 0$.
Conclude that ST acts transitively on $\mathbf{C}^{2n} \setminus 0$.

(c) Define (u,v) to be a hyperbolic pair if $\omega(u,v) = 1$. Show that ST acts transitively on hyperbolic pairs as follows. Given (u_1,v_1) and (u_2,v_2), there is a symplectic transvection T so that $T(u_1) = u_2$. If $\omega(v_2, T(v_1)) \neq 0$, construct another symplectic transvection T' such that $T'(u_2) = u_2$ and $T'(T(v_1)) = v_2$.
Otherwise, use $(u_2, u_2 + T(v_1))$ as an intermediate step using the previous case.

(d) Show by induction on n that $ST = \mathbf{Sp}_{2n}(\mathbf{C})$ as follows. Deduce the case $n = 1$ from the previous part.
For $n > 1$, given $g \in \mathbf{Sp}_{2n}(\mathbf{C})$, there is a symplectic transvection T such that $T(g(e_1)) = e_1$ and $T(g(e_{-1})) = e_{-1}$. Next, Tg acts on $\mathbf{C}^{2n-2} = \operatorname{span}(e_2, \dots, e_n, e_{-n}, \dots, e_{-2})$ and preserves its symplectic form. Let g' be the corresponding element of $\mathbf{Sp}_{2n-2}(\mathbf{C})$. By induction, g' is a product of symplectic transvections in $\mathbf{Sp}_{2n-2}(\mathbf{C})$.
Use this to show that g is a product of symplectic transvections in $\mathbf{Sp}_{2n}(\mathbf{C})$.

(e) Finally, if $g \in \mathbf{Sp}_{2n}(\mathbf{C})$, write it as a product $T_{a_1,\lambda_1} \cdots T_{a_r,\lambda_r}$. Define a function $\alpha_g \colon \mathbf{C} \to \mathbf{Sp}_{2n}(\mathbf{C})$ by $\alpha_g(t) = T_{a_1,t\lambda_1} \cdots T_{a_r,t\lambda_r}$. Show that α_g is continuous with respect to the Zariski topology. Conclude that g is in the same connected component as the identity matrix and hence $\mathbf{Sp}_{2n}(\mathbf{C})$ is connected.