Math 251C, Spring 2020
Homework 1

(1) (a) Let $\mathbf{C}^\times \cong \mathbf{GL}_1(\mathbf{C})$ be the group of nonzero complex numbers under multiplication. Show that every 1-dimensional rational representation $\rho\colon \mathbf{C}^\times \to \mathbf{C}^\times$ must be $\rho(z) = z^d$ for some integer d.
   (b) Conclude that every weight of a rational $\mathbf{GL}_n(\mathbf{C})$-representation is of the form $\mathrm{diag}(x_1, \dots, x_n) \mapsto x_1^{\mu_1} \cdots x_n^{\mu_n}$ for integers $\mu_1, \dots, \mu_n$ where $\mathrm{diag}(x_1, \dots, x_n)$ means diagonal matrix with the entries $x_1, \dots, x_n$.

(2) (a) A complete flag of V is a sequence of subspaces $V_1 \subset V_2 \subset \cdots \subset V_n = V$ where $\dim V_i = i$. Show that Borel subgroups $B \subset \mathbf{GL}(V)$ are in bijection with complete flags.
   (b) Show that maximal tori $T \subset \mathbf{GL}(V)$ are in bijection with direct sum decompositions $V = L_1 \oplus \cdots \oplus L_n$ where $\dim L_i = 1$.

(3) As representations of $\mathbf{GL}_n(\mathbf{C})$, find all of the highest weight vectors in $\mathbf{C}^n \otimes \mathbf{C}^n$ and $\mathbf{C}^n \otimes \mathbf{C}^n \otimes \mathbf{C}^n$.

(4) Let $U = \{(u,t) \mid u \in \mathbf{C}^{n\times n},\ t \in \mathbf{C}\}$ be the vector space of pairs consisting of an $n \times n$ matrix and complex number. Define an action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_n(\mathbf{C})$ on U by

$$(g,h) \cdot (u,t) = ((g^{-1})^T u h^{-1}, (\det g)(\det h)t).$$

Let $X = Z(t \det u - 1)$. Show that X is closed under the action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_n(\mathbf{C})$, that $\mathbf{C}[X]$ is a multiplicity-free representation, and determine which representations appear.

(5) Let $U = (\mathbf{C}^n \otimes \mathbf{C}^m)^*$ as in the example on generic matrices.
   (a) Define $X_{\le r} = \{u \in U \mid \mathrm{rank}\, u \le r\}$ where the rank is in the sense of matrices. Show that this is Zariski closed and that it is closed under the action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$. Show that these are all of the Zariski closed subsets closed under the action of $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$.
   (b) Determine the structure of the coordinate ring $\mathbf{C}[X_{\le r}]$ as a representation.
   (c) Let $I_{\le r}$ be the ideal of all polynomials in $\mathbf{C}[U]$ that are identically 0 on $X_{\le r}$. Determine the structure of $I_{\le r}$ as a representation and find generators for it as an ideal.
   (d) Repeat the first 3 steps when U is the space of symmetric matrices and when U is the space of skew-symmetric matrices.