

NOTES FOR MATH 847 (REPRESENTATION STABILITY)

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1. INTRODUCTION

1.1. **Prerequisites.** Prior familiarity with homological algebra will be useful but not essential. I'll review the key points as we go along, but will not give the proofs of basic facts. For details, any textbook on modern algebra should suffice.

This is an example-driven theory, and we will go through some examples from group theory, representation theory, topology, and algebraic geometry. I also will not assume these as prerequisites, and again will just state the facts that we'll need as they come up.

1.2. **Motivation.** We will study two aspects of representation stability in this course:

- A generalization of homological stability: one is given a sequence of abelian groups / vector spaces V_1, V_2, \dots together with homomorphisms $V_1 \xrightarrow{f_1} V_2 \xrightarrow{f_2} \dots$. These groups may arise as the (co)homology of some family of objects, and the phenomena of interest is when f_i are isomorphisms for $i \gg 0$.

As an example, one might consider the homology of symmetric groups Σ_n , and more specifically the first homology group. In general, for any group G , $H_1(G) = G/[G, G]$, the abelianization of G ($[G, G]$ is the subgroup of G generated by elements of the form $xyx^{-1}y^{-1}$). The standard inclusions $\Sigma_1 \rightarrow \Sigma_2 \rightarrow \dots$ induces maps $H_1(\Sigma_1) \rightarrow H_1(\Sigma_2) \rightarrow \dots$. While $H_1(\Sigma_1)$ is trivial, $H_1(\Sigma_i) \cong \mathbf{Z}/2$ for $i \geq 2$, and one can check f_i is an isomorphism for $i \geq 2$.

For another example, one might take X_n to be the space of unordered n -tuples of distinct points in the plane and consider its homology groups.

- A method for exploiting symmetry to prove uniformity statements across a family of examples. We'll develop this more later, but for an example in algebraic geometry, consider a tensor product of vector spaces $V_1 \otimes \cdots \otimes V_n$. Every element is a sum of simple tensors (those of the form $v_1 \otimes \cdots \otimes v_n$ for $v_i \in V_i$); define the rank of an element to be r if r is the least number of simple tensor summands needed. Define the border rank to be $\leq r$ if an element can be arbitrarily approximated by rank r tensors. When $n = 2$, this all reduces to the usual notion of rank for matrices.

In general, it is hard to determine the (border) rank of a tensor, but a theorem of Draisma–Kuttler that we will discuss says that one can test this by the vanishing of polynomials of degree at most $C(r)$, where $C(r)$ depends only on r , but not on n or $\dim V_i$.

Remark 1.2.1. Why is tensor rank interesting? Here's one example. Multiplication of $n \times n$ matrices can be represented by a bilinear map $\mathbf{k}^{n^2} \otimes \mathbf{k}^{n^2} \rightarrow \mathbf{k}^{n^2}$, which is the same as a tensor in $(\mathbf{k}^{n^2})^* \otimes (\mathbf{k}^{n^2})^* \otimes \mathbf{k}^{n^2}$. The rank of this tensor is the minimum number of multiplications (in \mathbf{k}) needed to perform matrix multiplication (see [La, §1.1, 1.2]). The naive algorithm gives an upper bound of n^3 , but one can do better. In particular, for $n = 2$, the rank is actually 7, while the precise value is already unknown for $n = 3$ (it's at least 19 and at most 23).

The formula is a bit complicated, but the idea can be illustrated with multiplication of complex numbers. The standard way gives four real multiplications:

$$(a + bi)(c + di) = (ac - bd) + (bc + ad)i.$$

Alternatively, we have

$$\begin{aligned} (a + bi)(c + di) &= (k_1 - k_3) + (k_1 + k_2)i \\ k_1 &= c(a + b), \quad k_2 = a(d - c), \quad k_3 = b(c + d), \end{aligned}$$

so you can get away with only 3 real multiplications at the cost of more additions. However, in some scenarios, minimizing the number of multiplications at the cost of increasing additions is desirable (for example, doing things by hand or with floating point approximation). \square

To elaborate on the first point, assume the V_i are finite-dimensional vector spaces over a field \mathbf{k} for simplicity. An obvious consequence of stability for the sequence $V_1 \rightarrow V_2 \rightarrow \cdots$ is that the sequence $(\dim V_i)_{i \geq 0}$ is eventually constant.

One of the first classes of examples of representation stable sequences is to consider a sequence V_i as before, where now V_i is a representation of the symmetric group Σ_i . The transition maps $f_i: V_i \rightarrow V_{i+1}$ should be Σ_i -equivariant, where we embed $\Sigma_i \subset \Sigma_{i+1}$ by having it fix $i + 1$, and they should also obey another condition which we omit for now.

The right generalization of stability is finite generation. For $i \leq j$, set $f_{j,i} = f_j \circ f_{j-1} \circ \cdots \circ f_i$. Say that x_1, \dots, x_d with $x_i \in V_{n_i}$ generate the sequence V if every $x \in V_n$ can be written as a $\mathbf{k}[\Sigma_n]$ -linear combination of the $f_{n,n_i}(x_i)$.

Example 1.2.2. (1) Let $V_n = \mathbf{k}^n$ with Σ_n acting by permutations on the coordinates.

The map $f_n: \mathbf{k}^n \rightarrow \mathbf{k}^{n+1}$ is the inclusion $(a_1, \dots, a_n) \mapsto (a_1, \dots, a_n, 0)$. The element $1 \in V_1$ generates the sequence.

- (2) Let $V_n = \mathbf{k}^{\binom{n}{2}}$ with a basis consisting of 2-element subsets of $\{1, \dots, n\}$ with the usual permutation action. The maps f_n are the obvious ones. Then $e_{\{1,2\}} \in V_2$ generates. More generally, we can do the same with k -element subsets of $\{1, \dots, n\}$.

- (3) Let $V_n = \mathbf{k}^{n^2}$ with a basis consisting of ordered pairs of elements in $\{1, \dots, n\}$ with the usual permutation action. Then $e_{(1,1)} \in V_1$ and $e_{(1,2)} \in V_2$ gives a generating set. \square

Finite generation implies a number of things about the representations V_i , but one consequence avoiding representation theory is that the sequence $(\dim V_i)_{i \geq 0}$ grows like a polynomial function for $i \gg 0$. This definition of sequences of Σ_n -representations is a bit clumsy, and we'll see later that the language of **FI**-modules will give a more natural notion to work with.

There are many sources of examples of these kinds of sequences. Related to the above example, we can take X_n to be the space of ordered n -tuples of distinct points in the plane and consider its homology groups. Each of these carries a natural action of the symmetric group Σ_n .

The representation theory of symmetric groups is a well-developed theory, especially over fields of characteristic 0. This will be the focus for the first part of the course.

2. REPRESENTATION THEORY

We won't go through most of the proofs for this part, just enough to get a feeling for how to work these objects. Some of this material is found in more detail in my symmetric functions notes:

<http://www.math.wisc.edu/~svs/740/notes.pdf>

2.1. Schur–Weyl duality. We will need to know that the irreducible complex representations of Σ_n are indexed by partitions of size n . There are several ways to get to this, but we will use Schur–Weyl duality to motivate the indexing.

Let A be a finite-dimensional algebra over a field \mathbf{k} . Recall that an A -module M is **simple** if its only submodules are 0 and M itself. An A -module M is **semisimple** if, for every submodule $N \subset M$, there exists another submodule $N' \subset M$ such that $N \cap N' = 0$ and $N + N' = M$, i.e., the smallest submodule containing both N and N' is M . We denote this relation by $M = N \oplus N'$. We say that A is semisimple if all of its finite-dimensional modules are semisimple.

Note that if M is semisimple and finite-dimensional, then we can decompose M as a direct sum of simple modules.

Let $M_n(\mathbf{k})$ be the algebra of $n \times n$ matrices with entries in \mathbf{k} . Given a module M in general, we let $\text{End}(M)$ denote the ring of linear endomorphisms (self-maps) of M . We won't prove the next statement, but it can be found in many abstract algebra textbooks.

Lemma 2.1.1. *For any n_1, \dots, n_r , the product $\prod_{i=1}^r M_{n_i}(\mathbf{k})$ is a semisimple algebra. If \mathbf{k} is algebraically closed, then every semisimple finite-dimensional algebra is of this form where r is the number of isomorphism classes of irreducible representations, and the n_i are their dimensions.*

Given a group G , the **group algebra** $\mathbf{k}[G]$ is the vector space with basis $\{e_g \mid g \in G\}$ with multiplication $e_g e_h = e_{gh}$. A $\mathbf{k}[G]$ -module is the same thing as a linear representation of G , so we will use these perspectives interchangeably.

Lemma 2.1.2. *Let \mathbf{k} be a field, G be a finite group, and suppose that $|G|$ is invertible in \mathbf{k} . Then $\mathbf{k}[G]$ is semisimple.*

Proof. Let M be a representation of G and let $N \subset M$ be a submodule. Choose any linear projection $\pi: M \rightarrow N$, i.e., $\pi(x) = x$ for all $x \in N$. Define a new map $\psi: M \rightarrow N$ by

$$\psi(m) = \frac{1}{|G|} \sum_{g \in G} g\psi(g^{-1}m).$$

Then

- $\psi(x) = x$ for all $x \in N$: given $x \in N$, $g^{-1}x \in N$ since N is a submodule, and $\psi(g^{-1}x) = g^{-1}x$, so the sum simplifies to x ,
- ψ is G -equivariant, i.e., $h\psi(m) = \psi(hm)$ for all $h \in G$ and $m \in M$:

$$h\psi(m) = \frac{1}{|G|} \sum_{g \in G} hg\psi(g^{-1}m) = \frac{1}{|G|} \sum_{g' \in G} g'\psi((g')^{-1}hm) = \psi(hm)$$

where in the second equality, we do the change of indexing $g' = hg$.

Let $N' = \ker \psi$. Then $M = N \oplus N'$. □

We will be concerned with $G = \Sigma_n$ the symmetric group, and $\mathbf{k} = \mathbf{C}$, the field of complex numbers.

Given an algebra A acting on a vector space E , let

$$B = \text{End}_A(E) = \{\varphi: E \rightarrow E \mid \varphi(ae) = a\varphi(e) \text{ for all } a \in A \text{ and } e \in E\}.$$

Proposition 2.1.3 (Double centralizer theorem). *Let E be a finite-dimensional vector space. Suppose $A \subset \text{End}(E)$ is a semisimple subalgebra.*

- (1) $A = \text{End}_B(E)$.
- (2) B is semisimple.
- (3) As a representation of $A \times B$, E decomposes as

$$E = \bigoplus_{i \in I} V_i \otimes W_i,$$

where the V_i are all of the irreducible A -modules and the W_i are all of the irreducible B -modules. In particular, the correspondence $V_i \leftrightarrow W_i$ is a bijection between the irreducible modules of A and B .

Proof. Since A is semisimple, we can decompose E into a direct sum of simple A -modules:

$$E = \bigoplus_{i \in I} V_i^{\oplus c_i} = \bigoplus_{i \in I} V_i \otimes W_i,$$

where $W_i = \text{Hom}_A(V_i, E)$ is the multiplicity space of V_i . Then $B = \bigoplus_{i \in I} \text{End}(W_i)$, so it is semisimple, and the simple B -modules are the W_i . It is also clear from this decomposition that $A = \bigoplus_{i \in I} \text{End}(V_i) = \text{End}_B(E)$, and so A is semisimple by Lemma 2.1.1. □

Now we come to our example of interest. Let $V = \mathbf{C}^n$ and let $E = V^{\otimes d}$. Then Σ_d acts on E by permuting the tensor factors, i.e.,

$$\sigma \cdot \sum v_1 \otimes \cdots \otimes v_d = \sum v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}.$$

In particular, this defines a homomorphism $\mathbf{C}[\Sigma_d] \rightarrow \text{End}(E)$, and we let A be the image. By definition, quotients of semisimple algebras are still semisimple, so A is semisimple by Lemma 2.1.2.

There is also an action of $\mathbf{GL}(V) = \mathbf{GL}_n(\mathbf{C})$ (the group of invertible linear transformations of V) by the natural action of linear change of coordinates, i.e.,

$$g \cdot \sum v_1 \otimes \cdots \otimes v_d = \sum (gv_1) \otimes \cdots \otimes (gv_d).$$

This gives a group homomorphism $\mathbf{GL}(V) \rightarrow \mathbf{GL}(E)$, and we let B be the algebra generated by the image. Evidently, we have $B \subseteq \text{End}_A(E)$ and $A \subseteq \text{End}_B(E)$.

Proposition 2.1.4 (Schur–Weyl duality). *$B = \text{End}_A(E)$. In particular, $A = \text{End}_B(E)$ and we have a decomposition*

$$E = \bigoplus_{i \in I} V_i \otimes W_i$$

where the V_i are all of the irreducible modules for A (and hence a subset of the irreducible representations of Σ_d) and the W_i are all of the irreducible modules for B (and hence a subset of the irreducible representations of $\mathbf{GL}(V)$).

Proof. We have an identification $\text{End}(V^{\otimes d}) = \text{End}(V)^{\otimes d}$, and $\text{End}_A(E)$ is identified with the symmetric elements in $\text{End}(V)^{\otimes d}$. By the next lemma, $\text{End}_A(E)$ is spanned by the elements $\varphi^{\otimes d} = \varphi \otimes \cdots \otimes \varphi$ where $\varphi \in \text{End}(V)$. We claim that $\varphi^{\otimes d} \in B$. Note that $t\text{id} + \varphi$ is invertible for all but finitely many values of t . In particular, $(t\text{id} + \varphi)^{\otimes d} \in B$. Linear spaces are closed (in the Euclidean topology, or Zariski topology if you prefer), so we can take the limit $t \rightarrow 0$ to conclude that $\varphi^{\otimes d} \in B$.

The remainder of the proposition follows from the double centralizer theorem. □

Lemma 2.1.5. *Let W be a vector space over a field of characteristic 0. Then the Σ_d -invariant elements in $W^{\otimes d}$ are linearly spanned by elements of the form $w^{\otimes d} = w \otimes \cdots \otimes w$ for $w \in W$.*

Proof. Pick a basis x_1, \dots, x_n for W . The subspace spanned by the $w^{\otimes d}$ is a $\mathbf{GL}(W)$ -invariant subspace of the symmetric invariants of $W^{\otimes d}$, which we can identify with the space $\text{Sym}^d(W)$ of homogeneous polynomials in variables x_1, \dots, x_n , so it suffices to show that any nonzero $\mathbf{GL}(W)$ -invariant subspace of $\text{Sym}^d(W)$ is the whole space. This has a standard basis consisting of monomials.

So let U be any $\mathbf{GL}(W)$ -invariant subspace of $\text{Sym}^d(W)$. First, consider a diagonal matrix in $\mathbf{GL}(W)$ with generic entries. Using a Vandermonde-type argument, one can show that if $f \in U$, then all of the monomials with nonzero coefficient in f are also in U . Now, given any monomial $x_1^{d_1} \cdots x_n^{d_n} \in U$, we can conclude that any other monomial is also in U . We illustrate this with an example. Let $g \in \mathbf{GL}(W)$ be the matrix which fixes x_2, \dots, x_n and sends x_1 to $x_1 + x_2$. Then $(x_1 + x_2)^{d_1} x_2^{d_2} \cdots x_n^{d_n} \in U$ and $x_1^{d_1-1} x_2^{d_2+1} \cdots x_n^{d_n}$ has a nonzero coefficient (this is where characteristic 0 is important) and hence also belongs to U . By lowering and raising exponents appropriately, we can get to any other monomial. □

Remark 2.1.6. The statement about $\text{Sym}^d(W)$ is not correct in positive characteristic. For example, let $W = \mathbf{k}^n$ for a field of characteristic p . Then the space spanned by x_1^p, \dots, x_n^p in $\text{Sym}^p(W)$ is closed under $\mathbf{GL}(W)$. However, it cannot be identified with the subspace of Σ_p -invariants in $W^{\otimes p}$ in a $\mathbf{GL}(W)$ -equivariant way. This subspace of Σ_p -invariants is also known as the divided power of W . With some more care, the statement about divided powers can be proven in more generality, but we won't need it. □

The next question is to identify the indexing set I in the decomposition given by Schur–Weyl duality. We’ll use some information about $\mathbf{GL}_n(\mathbf{C})$ to do this, and go via symmetric functions.

2.2. Symmetric functions. Let x_1, \dots, x_n be a finite set of indeterminates. The symmetric group on n letters is Σ_n . It acts on $\mathbf{Z}[x_1, \dots, x_n]$, the ring of polynomials in n variables and integer coefficients, by substitution of variables:

$$\sigma \cdot f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The **ring of symmetric polynomials** is the set of fixed polynomials:

$$\Lambda(n) := \{f \in \mathbf{Z}[x_1, \dots, x_n] \mid \sigma \cdot f = f \text{ for all } \sigma \in \Sigma_n\}.$$

This is a subring of $\mathbf{Z}[x_1, \dots, x_n]$.

We will also treat the case $n = \infty$. Let x_1, x_2, \dots be a countably infinite set of indeterminates. Let Σ_∞ be the group of all permutations of $\{1, 2, \dots\}$. Let R be the ring of power series in x_1, x_2, \dots of *bounded degree*. Hence, elements of R can be infinite sums, but only in a finite number of degrees.

Write $\pi_n: \Lambda \rightarrow \Lambda(n)$ for the homomorphism which sets $x_{n+1} = x_{n+2} = \dots = 0$.

Remark 2.2.1. (For those familiar with inverse limits.) There is a ring homomorphism $\pi_{n+1,n}: \Lambda(n+1) \rightarrow \Lambda(n)$ obtained by setting $x_{n+1} = 0$. Furthermore, $\Lambda(n) = \bigoplus_{d \geq 0} \Lambda(n)_d$ where $\Lambda(n)_d$ is the subgroup of homogeneous symmetric polynomials of degree d . The map $\pi_{n+1,n}$ restricts to a map $\Lambda(n+1)_d \rightarrow \Lambda(n)_d$; set

$$\Lambda_d = \varprojlim_n \Lambda(n)_d.$$

Then

$$\Lambda = \bigoplus_{d \geq 0} \Lambda_d.$$

Note that we aren’t saying that Λ is the inverse limit of the $\Lambda(n)$; the latter object includes infinite sums of unbounded degree. \square

Then Σ_∞ acts on R , and we define the **ring of symmetric functions**

$$\Lambda := \{f \in R \mid \sigma \cdot f = f \text{ for all } \sigma \in \Sigma_\infty\}.$$

Again, this is a subring of R .

Example 2.2.2. Here are some basic examples of elements in Λ (we will study them more soon):

$$\begin{aligned} p_k &:= \sum_{i \geq 1} x_i^k \\ e_k &:= \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \cdots x_{i_k} \\ h_k &:= \sum_{i_1 \leq i_2 \leq \dots \leq i_k} x_{i_1} x_{i_2} \cdots x_{i_k}. \end{aligned} \quad \square$$

Sometimes, we want to work with rational coefficients instead of integer coefficients. In that case, we’ll write $\Lambda_{\mathbf{Q}}$ or $\Lambda(n)_{\mathbf{Q}}$ to denote the appropriate rings.

2.3. Polynomial representations of general linear groups. Let $\mathbf{GL}_n(\mathbf{C})$ denote the group of invertible $n \times n$ complex matrices.

A **polynomial representation** of $\mathbf{GL}_n(\mathbf{C})$ is a homomorphism $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(V)$ where V is a \mathbf{C} -vector space, and the entries of ρ can be expressed in terms of polynomials (as soon as we pick a basis for V).

A simple example is the identity map $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}_n(\mathbf{C})$. Slightly more sophisticated is $\rho: \mathbf{GL}_2(\mathbf{C}) \rightarrow \mathbf{GL}(\text{Sym}^2(\mathbf{C}^2))$ where $\text{Sym}^2(\mathbf{C}^2)$ is the space of degree 2 polynomials in x, y (which is a basis for \mathbf{C}^2). The homomorphism can be defined by linear change of coordinates, i.e.,

$$\rho(g)(ax^2 + bxy + cy^2) = a(gx)^2 + b(gx)(gy) + c(gy)^2.$$

If we pick the basis x^2, xy, y^2 for $\text{Sym}^2(\mathbf{C}^2)$, this can be written in coordinates as

$$(2.3.1) \quad \mathbf{GL}_2(\mathbf{C}) \rightarrow \mathbf{GL}_3(\mathbf{C})$$

$$\begin{pmatrix} g_{1,1} & g_{1,2} \\ g_{2,1} & g_{2,2} \end{pmatrix} \mapsto \begin{pmatrix} g_{1,1}^2 & g_{1,1}g_{1,2} & g_{1,2}^2 \\ 2g_{1,1}g_{2,1} & g_{1,1}g_{2,2} + g_{1,2}g_{2,1} & 2g_{1,2}g_{2,2} \\ g_{2,1}^2 & g_{2,1}g_{2,2} & g_{2,2}^2 \end{pmatrix}.$$

More generally, we can define $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(\text{Sym}^d(\mathbf{C}^n))$ for any n, d . Another important example uses exterior powers instead of symmetric powers, so we have $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(\wedge^d(\mathbf{C}^n))$.

An important invariant of a polynomial representation ρ is its **character**: define

$$\text{char}(\rho)(x_1, \dots, x_n) := \text{Tr}(\rho(\text{diag}(x_1, \dots, x_n))),$$

where $\text{diag}(x_1, \dots, x_n)$ is the diagonal matrix with entries x_1, \dots, x_n and Tr denotes trace.

Lemma 2.3.2. $\text{char}(\rho)(x_1, \dots, x_n) \in \Lambda(n)$.

Proof. Each permutation $\sigma \in \Sigma_n$ corresponds to a permutation matrix $M(\sigma)$: this is the matrix with a 1 in row $\sigma(i)$ and column i for $i = 1, \dots, n$ and 0's everywhere else. Then

$$M(\sigma)^{-1} \text{diag}(x_1, \dots, x_n) M(\sigma) = \text{diag}(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

Now use that the trace of a matrix is invariant under conjugation:

$$\begin{aligned} \text{char}(\rho)(x_1, \dots, x_n) &= \text{Tr}(\rho(\text{diag}(x_1, \dots, x_n))) \\ &= \text{Tr}(\rho(M(\sigma))^{-1} \rho(\text{diag}(x_1, \dots, x_n)) \rho(M(\sigma))) \\ &= \text{Tr}(\rho(M(\sigma)^{-1} \text{diag}(x_1, \dots, x_n) M(\sigma))) \\ &= \text{Tr}(\rho(\text{diag}(x_{\sigma(1)}, \dots, x_{\sigma(n)}))) \\ &= \text{char}(\rho)(x_{\sigma(1)}, \dots, x_{\sigma(n)}). \end{aligned} \quad \square$$

Example 2.3.3. • The character of the identity representation is $x_1 + x_2 + \dots + x_n$.

• The character of the representation $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(\text{Sym}^d(\mathbf{C}^n))$ is

$$h_n(x_1, \dots, x_n) = \sum_{1 \leq i_1 \leq \dots \leq i_d \leq n} x_{i_1} \cdots x_{i_d}.$$

• The character of the representation $\rho: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(\wedge^d(\mathbf{C}^n))$ is

$$e_n(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_d \leq n} x_{i_1} \cdots x_{i_d}. \quad \square$$

A few remarks that aren't easy to see right now (though we may revisit):

- The set of characters in $\Lambda(n)$ generates all of $\Lambda(n)$ as an abelian group.
- If we are more careful about how to define characters in infinite-dimensional settings, we get that characters of polynomial representations of $\mathbf{GL}_\infty(\mathbf{C})$ are elements of Λ .
- The character determines the representation up to isomorphism: if $\text{char}(\rho) = \text{char}(\rho')$, then ρ and ρ' define isomorphic representations: one can be obtained from the other by a suitable isomorphism of the underlying vector spaces V and V' .

If we take these remarks as fact for now, this gives one motivation for studying Λ . This representation-theoretic interpretation of Λ will clarify various definitions and constructions we will encounter. A few basic ones that we can see now:

- If $\rho_i: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(V_i)$ are polynomial representations for $i = 1, 2$, we can form the direct sum representation $\rho_1 \oplus \rho_2: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(V_1 \oplus V_2)$ via

$$(\rho_1 \oplus \rho_2)(g) = \begin{pmatrix} \rho_1(g) & 0 \\ 0 & \rho_2(g) \end{pmatrix}$$

and

$$\text{char}(\rho_1 \oplus \rho_2)(x_1, \dots, x_n) = \text{char}(\rho_1)(x_1, \dots, x_n) + \text{char}(\rho_2)(x_1, \dots, x_n).$$

- There's also a multiplicative version using tensor product. If $\rho_i: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(V_i)$ are polynomial representations for $i = 1, 2$, we can form the tensor product representation $\rho_1 \otimes \rho_2: \mathbf{GL}_n(\mathbf{C}) \rightarrow \mathbf{GL}(V_1 \otimes V_2)$ via (assuming $\rho_1(g)$ is $N \times N$):

$$(\rho_1 \otimes \rho_2)(g) = \begin{pmatrix} \rho_1(g)_{1,1}\rho_2(g) & \rho_1(g)_{1,2}\rho_2(g) & \cdots & \rho_1(g)_{1,N}\rho_2(g) \\ \rho_1(g)_{2,1}\rho_2(g) & \rho_1(g)_{2,2}\rho_2(g) & \cdots & \rho_1(g)_{2,N}\rho_2(g) \\ \vdots & \vdots & \ddots & \vdots \\ \rho_1(g)_{N,1}\rho_2(g) & \rho_1(g)_{N,2}\rho_2(g) & \cdots & \rho_1(g)_{N,N}\rho_2(g) \end{pmatrix}$$

(here we are multiplying $\rho_2(g)$ by each entry of $\rho_1(g)$ and creating a giant block matrix) and

$$\text{char}(\rho_1 \otimes \rho_2)(x_1, \dots, x_n) = \text{char}(\rho_1)(x_1, \dots, x_n) \cdot \text{char}(\rho_2)(x_1, \dots, x_n).$$

Note that subtraction will not have any natural interpretation, and in general, the difference of two characters need not be a character. In general, the elements of $\Lambda(n)$ or Λ can be thought of as virtual characters since every element is the difference of two characters.

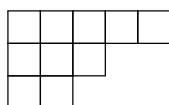
2.4. Partitions. A **partition** of a nonnegative integer n is a sequence $\lambda = (\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ and $\lambda_1 + \cdots + \lambda_k = n$. We will consider two partitions the same if their nonzero entries are the same. And for shorthand, we may omit the commas, so the partition $(1, 1, 1, 1)$ of 4 can be written as 1111. As a further shorthand, the exponential notation is used for repetition, so for example, 1^4 is the partition $(1, 1, 1, 1)$. We let $\text{Par}(n)$ be the set of partitions of n , and denote the size by $p(n) = |\text{Par}(n)|$. By convention, $\text{Par}(0)$ consists of exactly one partition, the empty one.

Example 2.4.1.

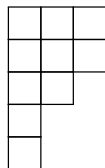
$$\begin{aligned} \text{Par}(1) &= \{1\}, \\ \text{Par}(2) &= \{2, 1^2\}, \\ \text{Par}(3) &= \{3, 21, 1^3\}, \\ \text{Par}(4) &= \{4, 31, 22, 21^2, 1^4\}, \\ \text{Par}(5) &= \{5, 41, 32, 31^2, 2^21, 21^3, 1^5\}. \end{aligned} \quad \square$$

If λ is a partition of n , we write $|\lambda| = n$ (**size**). Also, $\ell(\lambda)$ is the number of nonzero entries of λ (**length**). For each i , $m_i(\lambda)$ is the number of entries of λ that are equal to i .

It will often be convenient to represent partitions graphically. This is done via **Young diagrams**, which is a collection of left-justified boxes with λ_i boxes in row i .¹ For example, the Young diagram



corresponds to the partition $(5, 3, 2)$. Flipping across the main diagonal gives another partition λ^\dagger , called the **transpose**. In our example, flipping gives



So $(5, 3, 2)^\dagger = (3, 3, 2, 1, 1)$. In other words, the role of columns and rows has been interchanged. This is an important involution of $\text{Par}(n)$ which we will use later.

We will use several different partial orderings of partitions:

- $\lambda \subseteq \mu$ if $\lambda_i \leq \mu_i$ for all i .
- The **dominance order**: $\lambda \leq \mu$ if $\lambda_1 + \dots + \lambda_i \leq \mu_1 + \dots + \mu_i$ for all i . Note that if $|\lambda| = |\mu|$, then $\lambda \leq \mu$ if and only if $\lambda^\dagger \geq \mu^\dagger$. So transpose is an order-reversing involution on the set of partitions of a fixed size.
- The **lexicographic order**: for partitions of the same size, $\lambda \leq^R \mu$ if $\lambda = \mu$, or otherwise, there exists i such that $\lambda_1 = \mu_1, \dots, \lambda_{i-1} = \mu_{i-1}$, and $\lambda_i < \mu_i$. This is a total ordering.

2.5. Bases for Λ .

2.5.1. Monomial symmetric functions. Given a partition $\lambda = (\lambda_1, \lambda_2, \dots)$, define the **monomial symmetric function** by

$$m_\lambda = \sum_{\alpha} x^\alpha$$

where the sum is over all *distinct* permutations α of λ . This is symmetric by definition. So for example, $m_1 = \sum_{i \geq 1} x_i$ since all of the distinct permutations of $(1, 0, 0, \dots)$ are integer

¹In the **English convention**, row i sits above row $i + 1$, in the **French convention**, it is reversed. There is also the **Russian convention**, which is obtained from the French convention by rotating by 45 degrees counter-clockwise.

sequences with a single 1 somewhere and 0 elsewhere. By convention, $m_0 = 1$. Some other examples:

$$m_{1,1} = \sum_{i < j} x_i x_j$$

$$m_{3,2,1} = \sum_{\substack{i,j,k \\ i \neq j, j \neq k, i \neq k}} x_i x_j x_k.$$

In general, $m_{1^k} = e_k$ and $m_k = p_k$.

Theorem 2.5.1. *As we range over all partitions, the m_λ form a basis for Λ .*

Proof. They are linearly independent since no two m_λ have any monomials in common. Clearly they also span: given $f \in \Lambda$, we can write $f = \sum_\alpha c_\alpha x^\alpha$ and $c_\alpha = c_\beta$ if both are permutations of each other, so this can be rewritten as $f = \sum_\lambda c_\lambda m_\lambda$ where the sum is now over just the partitions. \square

Corollary 2.5.2. Λ_d has a basis given by $\{m_\lambda \mid |\lambda| = d\}$, and hence is a free abelian group of rank $p(d) = |\text{Par}(d)|$.

Theorem 2.5.3. $\Lambda(n)_d$ has a basis given by $\{m_\lambda(x_1, \dots, x_n) \mid |\lambda| = d, \ell(\lambda) \leq n\}$.

2.5.2. *Elementary symmetric functions.* Recall that we defined

$$e_k = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1} x_{i_2} \dots x_{i_k}.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, define the **elementary symmetric function** by

$$e_\lambda = e_{\lambda_1} e_{\lambda_2} \dots e_{\lambda_k}.$$

Note $e_\lambda \in \Lambda_{|\lambda|}$.

Theorem 2.5.4. *The e_λ form a basis of Λ .*

Theorem 2.5.5. *The set $\{e_\lambda(x_1, \dots, x_n) \mid \lambda_1 \leq n, |\lambda| = d\}$ is a basis of $\Lambda(n)_d$.*

Proof. If $\lambda_1 > n$, then $e_{\lambda_1}(x_1, \dots, x_n) = 0$, so $e_\lambda(x_1, \dots, x_n) = 0$. Hence under the map $\pi_n: \Lambda \rightarrow \Lambda(n)$, the proposed e_λ span the image. The number of such e_λ in degree d is $|\{\lambda \mid \lambda_1 \leq n, |\lambda| = d\}|$, which is the same as $|\{\lambda \mid \ell(\lambda) \leq n, |\lambda| = d\}|$ via the transpose \dagger , and this is the rank of $\Lambda(n)_d$, so the e_λ form a basis. \square

Remark 2.5.6. The previous two theorems say that the elements e_1, e_2, e_3, \dots are algebraically independent in Λ , and that the elements e_1, \dots, e_n are algebraically independent in $\Lambda(n)$. This is also known as the “fundamental theorem of symmetric functions”. \square

2.5.3. *Complete homogeneous symmetric functions.* For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, define the **complete homogeneous symmetric functions** by

$$h_\lambda = h_{\lambda_1} \dots h_{\lambda_k}.$$

Theorem 2.5.7. *The h_λ form a basis for Λ .*

Theorem 2.5.8. h_1, \dots, h_n are algebraically independent generators of $\Lambda(n)$, and the set $\{h_\lambda(x_1, \dots, x_n) \mid \lambda_1 \leq n, |\lambda| = d\}$ is a basis of $\Lambda(n)_d$.

2.5.4. *Power sum symmetric functions.* Recall we defined

$$p_k = \sum_{n \geq 1} x_n^k.$$

For a partition $\lambda = (\lambda_1, \dots, \lambda_k)$, the **power sum symmetric functions** are defined by

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}.$$

Theorem 2.5.9. *The p_λ are linearly independent.*

Remark 2.5.10. The p_λ do not form a basis for Λ . For example, in degree 2, we have

$$\begin{aligned} p_2 &= m_2 \\ p_{1,1} &= m_2 + 2m_{1,1} \end{aligned}$$

and the change of basis matrix has determinant 2, so is not invertible over \mathbf{Z} . However, they do form a basis for $\Lambda_{\mathbf{Q}}$. \square

Theorem 2.5.11. *p_1, \dots, p_n are algebraically independent generators of $\Lambda(n)_{\mathbf{Q}}$ and the set $\{p_\lambda(x_1, \dots, x_n) \mid \lambda_1 \leq n, |\lambda| = d\}$ is a basis for $\Lambda(n)_{\mathbf{Q},d}$.*

2.5.5. *The involution ω .* Since the e_i are algebraically independent, we can define a ring homomorphism $f: \Lambda \rightarrow \Lambda$ by specifying $f(e_i)$ arbitrarily.² Define

$$\omega: \Lambda \rightarrow \Lambda$$

by $\omega(e_i) = h_i$, where recall that $h_k = \sum_{i_1 \leq \dots \leq i_k} x_{i_1} \cdots x_{i_k}$.

Theorem 2.5.12. *ω is an involution, i.e., $\omega^2 = 1$. Equivalently, $\omega(h_i) = e_i$.*

Furthermore, we can define a finite analogue of ω , the ring homomorphism $\omega_n: \Lambda(n) \rightarrow \Lambda(n)$, given by $\omega_n(e_i) = h_i$ for $i = 1, \dots, n$.

Theorem 2.5.13. *$\omega_n^2 = 1$, and ω_n is invertible. Equivalently, $\omega_n(h_i) = e_i$ for $i = 1, \dots, n$.*

2.5.6. *A scalar product.* Define a bilinear form $\langle \cdot, \cdot \rangle: \Lambda \otimes \Lambda \rightarrow \mathbf{Z}$ by setting

$$\langle m_\lambda, h_\mu \rangle = \delta_{\lambda,\mu}$$

where δ is the Kronecker delta (1 if $\lambda = \mu$ and 0 otherwise). In other words, if $f = \sum_\lambda a_\lambda m_\lambda$ and $g = \sum_\mu b_\mu h_\mu$, then $\langle f, g \rangle = \sum_\lambda a_\lambda b_\lambda$ (well-defined since both m and h are bases). At this point, the definition looks completely unmotivated. However, this inner product is natural from the representation-theoretic perspective, which we'll mention in the next section (without proof).

In our setup, m and h are dual bases with respect to the pairing. We will want a general criteria for two bases to be dual to each other. To state this criterion, we need to work in two sets of variables x and y and in the ring $\Lambda \otimes \Lambda$ where the x 's and y 's are separately symmetric.

Lemma 2.5.14. *Let u_λ and v_μ be bases of Λ (or $\Lambda_{\mathbf{Q}}$). Then $\langle u_\lambda, v_\mu \rangle = \delta_{\lambda,\mu}$ if and only if*

$$\sum_\lambda u_\lambda(x) v_\lambda(y) = \prod_{i,j} (1 - x_i y_j)^{-1}.$$

²Every element is uniquely of the form $\sum_\lambda c_\lambda e_\lambda$; since f is a ring homomorphism, it sends this to $\sum_\lambda c_\lambda f(e_{\lambda_1}) f(e_{\lambda_2}) \cdots f(e_{\lambda_{\ell(\lambda)}})$.

Corollary 2.5.15. *The pairing is symmetric, i.e., $\langle f, g \rangle = \langle g, f \rangle$.*

Proof. The condition above is the same if we interchange x and y , so $\langle m_\lambda, h_\mu \rangle = \langle h_\mu, m_\lambda \rangle$. Now use bilinearity. \square

Proposition 2.5.16.

$$\begin{aligned} \prod_{i,j} (1 - x_i y_j)^{-1} &= \sum_{\lambda} m_{\lambda}(x) h_{\lambda}(y) \\ &= \sum_{\lambda} z_{\lambda}^{-1} p_{\lambda}(x) p_{\lambda}(y). \end{aligned}$$

Proposition 2.5.17. *ω is an isometry, i.e., $\langle f, g \rangle = \langle \omega(f), \omega(g) \rangle$.*

The bilinear form $\langle \cdot, \cdot \rangle$ is positive definite, i.e., $\langle f, f \rangle > 0$ for $f \neq 0$.

2.5.7. *Schur functions.* Let λ be a partition. A **semistandard Young tableaux (SSYT)** T is an assignment of natural numbers to the Young diagram of λ so that the numbers are weakly increasing going left to right in each row, and the numbers are strictly increasing going top to bottom in each column.

Example 2.5.18. If $\lambda = (4, 3, 1)$, and we have the assignment

a	b	c	d
e	f	g	
h			

then, in order for this to be a SSYT, we need to have

- $a \leq b \leq c \leq d$,
- $e \leq f \leq g$,
- $a < e < h$,
- $b < f$, and
- $c < g$.

An example of a SSYT is

1	1	3	5
2	3	4	
7			

. \square

The type of a SSYT T is the sequence $\text{type}(T) = (\alpha_1, \alpha_2, \dots)$ where α_i is the number of times that i appears in T . We set

$$x^T = x_1^{\alpha_1} x_2^{\alpha_2} \dots$$

Given a pair of partitions $\mu \subseteq \lambda$, the Young diagram of λ/μ is the Young diagram of λ with the Young diagram of μ removed. We define a SSYT of shape λ/μ to be an assignment of natural numbers of this Young diagram which is weakly increasing in rows and strictly increasing in columns.

Example 2.5.19. If $\lambda = (5, 3, 1)$ and $\mu = (2, 1)$, then

		a	b	c
	d	e		
f				

is a SSYT if

- $a \leq b \leq c$,
- $d \leq e$, and

- $a < e$. □

We define the type of T and x^T in the same way.

Given a partition λ , the **Schur function** s_λ is defined by

$$s_\lambda = \sum_T x^T$$

where the sum is over all SSYT of shape λ . Similarly, given $\mu \subseteq \lambda$, the **skew Schur function** $s_{\lambda/\mu}$ is defined by

$$s_{\lambda/\mu} = \sum_T x^T$$

where the sum is over all SSYT of shape λ/μ . Note that this is a strict generalization of the first definition since we can take $\mu = \emptyset$, the unique partition of 0.

We can make the same definitions in finitely many variables x_1, \dots, x_n if we restrict the sums to be only over SSYT that only use the numbers $1, \dots, n$.

Example 2.5.20. $s_{1,1}(x_1, x_2, \dots, x_n)$ is the sum over SSYT of shape $(1, 1)$. This is the same as a choice of $1 \leq i < j \leq n$, so $s_{1,1}(x_1, \dots, x_n) = \sum_{1 \leq i < j \leq n} x_i x_j = e_2(x_1, \dots, x_n)$, and by the same reasoning, $s_{1,1} = e_2$ in infinitely many variables. More generally, $s_{1^k} = e_k$ for any k .

Also, $s_k = h_k$ since a SSYT of shape (k) is a choice of $i_1 \leq i_2 \leq \dots \leq i_k$.

For something different, consider $s_{2,1}(x_1, x_2, x_3)$. There are 8 SSYT that of shape $(2, 1)$ that only use 1, 2, 3:

$$\begin{array}{|c|c|} \hline 1 & 1 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 2 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 1 & 3 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 3 & \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline 2 & 3 \\ \hline 3 & \\ \hline \end{array}$$

From this, we can read off that $s_{2,1}(x_1, x_2, x_3)$ is a symmetric polynomial. Furthermore, it is $m_{2,1}(x_1, x_2, x_3) + 2m_{1,1,1}(x_1, x_2, x_3)$. □

Theorem 2.5.21. *For any $\mu \subseteq \lambda$, the skew Schur function $s_{\lambda/\mu}$ is a symmetric function.*

We now focus on Schur functions. Suppose λ is a partition of n . Let $K_{\lambda,\alpha}$ be the number of SSYT of shape λ and type α , this is called a **Kostka number**. The previous theorem says $K_{\lambda,\alpha} = K_{\lambda,\sigma(\alpha)}$ for any permutation σ , so it's enough to study the case when α is a partition. By the definition of Schur function, we have

$$s_\lambda = \sum_{\mu \vdash n} K_{\lambda,\mu} m_\mu.$$

An important special case is when $\mu = 1^n$. Then $K_{\lambda,1^n}$ is the number of SSYT that use each of the numbers $1, \dots, n$ exactly once. Such a SSYT is called a **standard Young tableau**, and $K_{\lambda,1^n}$ is denoted f^λ .

Theorem 2.5.22. *If $K_{\lambda,\mu} \neq 0$, then $\mu \leq \lambda$ (dominance order). Also, $K_{\lambda,\lambda} = 1$. In particular, the s_λ form a basis for Λ .*

Corollary 2.5.23. $\{s_\lambda \mid |\lambda| = d\}$ is a basis for Λ_d .

Corollary 2.5.24. $\{s_\lambda(x_1, \dots, x_n) \mid |\lambda| = d, \ell(\lambda) \leq n\}$ is a basis for $\Lambda(n)_d$.

Proof. Note that if $\ell(\lambda) > n$, there are no SSYT only using $1, \dots, n$, so $s_\lambda(x_1, \dots, x_n) = 0$. Hence the set in question spans $\Lambda(n)_d$. Since $\Lambda(n)_d$ is free of rank equal to the size of this set, it must also be a basis. □

2.6. Schur functors. For the material in this section, see [Wey, Chapter 2]. What we call \mathbf{S}_λ is denoted by $\mathbf{L}_{\lambda'}$ there.

Definition 2.6.1. Let R be a commutative ring and E a free R -module. Let λ be a partition with n parts and write $m = \lambda_1$. We use $S^n E$ to denote the n th symmetric power of E . The **Schur functor** $\mathbf{S}_\lambda(E)$ is the image of the map

$$\bigwedge^{\lambda_1^\dagger} E \otimes \cdots \otimes \bigwedge^{\lambda_m^\dagger} E \xrightarrow{\Delta} E^{\otimes \lambda_1^\dagger} \otimes \cdots \otimes E^{\otimes \lambda_m^\dagger} = E^{\otimes \lambda_1} \otimes \cdots \otimes E^{\otimes \lambda_n} \xrightarrow{\mu} S^{\lambda_1} E \otimes \cdots \otimes S^{\lambda_n} E,$$

where the maps are defined as follows. First, Δ is the product of the comultiplication maps $\bigwedge^i E \rightarrow E^{\otimes i}$ given by

$$e_1 \wedge \cdots \wedge e_i \mapsto \sum_{w \in \Sigma_i} \text{sgn}(w) e_{w(1)} \otimes \cdots \otimes e_{w(i)}.$$

The equals sign is interpreted as follows: pure tensors in $E^{\otimes \lambda_1^\dagger} \otimes \cdots \otimes E^{\otimes \lambda_m^\dagger}$ can be interpreted as filling the Young diagram of λ with vectors along the columns, which can be thought of as pure tensors in $E^{\otimes \lambda_1} \otimes \cdots \otimes E^{\otimes \lambda_n}$ by reading via rows. Finally, μ is the multiplication map $E^{\otimes i} \rightarrow S^i E$ given by $e_1 \otimes \cdots \otimes e_i \mapsto e_1 \cdots e_i$.

In particular, note that $\mathbf{S}_\lambda E = 0$ if the number of parts of λ exceeds $\text{rank } E$. \square

Example 2.6.2. Take $\lambda = (3, 2)$. Then the map is given by

$$\begin{aligned} (e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \otimes e_5 &\mapsto \begin{array}{|c|c|c|} \hline e_1 & e_3 & e_5 \\ \hline e_2 & e_4 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline e_2 & e_3 & e_5 \\ \hline e_1 & e_4 & \\ \hline \end{array} - \begin{array}{|c|c|c|} \hline e_1 & e_4 & e_5 \\ \hline e_2 & e_3 & \\ \hline \end{array} + \begin{array}{|c|c|c|} \hline e_2 & e_4 & e_5 \\ \hline e_1 & e_3 & \\ \hline \end{array} \\ &\mapsto (e_1 e_3 e_5 \otimes e_2 e_4) - (e_2 e_3 e_5 \otimes e_1 e_4) \\ &\quad - (e_1 e_4 e_5 \otimes e_2 e_3) + (e_2 e_4 e_5 \otimes e_1 e_3) \end{aligned} \quad \square$$

The construction of $\mathbf{S}_\lambda E$ is functorial with respect to E : given an R -linear map $f: E \rightarrow F$, we get an R -linear map $\mathbf{S}_\lambda(f): \mathbf{S}_\lambda E \rightarrow \mathbf{S}_\lambda F$ such that $\mathbf{S}_\lambda(f \circ g) = \mathbf{S}_\lambda(f) \circ \mathbf{S}_\lambda(g)$. This has two consequences: $\mathbf{S}_\lambda E$ is naturally a representation of $\mathbf{GL}(E)$ (in fact, it is more special because the action is defined for all linear operators of E , not just the invertible ones), and we can also construct $\mathbf{S}_\lambda \mathcal{E}$ when \mathcal{E} is a vector bundle.

Fix a basis e_1, \dots, e_n for E . Given a tableau T , we get an element e_T in $\bigwedge^{\lambda_1^\dagger} E \otimes \cdots \otimes \bigwedge^{\lambda_m^\dagger} E$ by taking the tensor product of the wedge products of the standard basis vectors coming from the entries in each column. For example, if $T = \begin{array}{|c|c|c|} \hline 1 & 3 & 5 \\ \hline 2 & 4 & \\ \hline \end{array}$ then $e_T = (e_1 \wedge e_2) \otimes (e_3 \wedge e_4) \otimes e_5$.

Definition 2.6.3. Given a box $b = (i, j) \in \lambda$, its **content** is $c(b) = j - i$ and its **hook length** is $h(b) = \lambda_i - i + \lambda_j^\dagger - j + 1$. \square

Example 2.6.4. Let $\lambda = (4, 3, 1)$. Then $\lambda^\dagger = (3, 2, 2, 1)$. The contents and hook lengths are given as follows:

$$c : \begin{array}{|c|c|c|c|} \hline 0 & 1 & 2 & 3 \\ \hline -1 & 0 & 1 & \\ \hline -2 & & & \\ \hline \end{array} \quad h : \begin{array}{|c|c|c|c|} \hline 6 & 4 & 3 & 1 \\ \hline 4 & 2 & 1 & \\ \hline 1 & & & \\ \hline \end{array} \quad \square$$

Theorem 2.6.5. The Schur functor $\mathbf{S}_\lambda E$ is a free R -module. If $\text{rank } E = n$, then

$$\text{rank } \mathbf{S}_\lambda E = \prod_{b \in \lambda} \frac{n + c(b)}{h(b)}.$$

A basis for $\mathbf{S}_\lambda E$ is given by the images of the e_T as T ranges over all SSYT of shape λ using the numbers $1, \dots, n$.

If R is a field, then the weight of e_T is x^T , and hence the character of $\mathbf{S}_\lambda E$ is the Schur function $s_\lambda(x_1, \dots, x_n)$.

The point is that to calculate things like tensor products of Schur functors, it suffices to understand how to multiply Schur functions. This is most useful in characteristic 0, where we have mentioned that the character completely encodes all information about the representation. We will discuss this in more detail later.

Remark 2.6.6. The construction we gave is unmotivated, but we can at least explain the indexing using generalities on irreducible representations of complex reductive groups, such as $\mathbf{GL}_n(\mathbf{C})$. A Borel subgroup B is a maximal (Zariski) closed, connected, solvable (algebraic) subgroup. For $\mathbf{GL}_n(\mathbf{C})$, one can take the subgroup of upper-triangular matrices. Fixing B and an irreducible representation V , there is a unique, up to scalar, nonzero vector $v \in V$ (**highest weight vector**) and an (algebraic) group homomorphism $\lambda: B \rightarrow \mathbf{C}^\times$ (**highest weight**) such that $b \cdot v = \lambda(b)v$ for all $b \in B$.

For the upper-triangular matrices, an algebraic group homomorphism $\lambda: B \rightarrow \mathbf{C}^\times$ takes the form $\lambda(b) = b_1^{\lambda_1} \cdots b_n^{\lambda_n}$ where b_1, \dots, b_n are the diagonal entries and $\lambda_i \in \mathbf{Z}$, so we can identify them with elements of \mathbf{Z}^n . It turns out that highest weights satisfy an additional constraint $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. The indexing for the Schur functors $\mathbf{S}_\lambda E$ is chosen so that it has highest weight λ . \square

2.7. Pieri's rule. Since the s_λ are a basis, we have unique expressions

$$(2.7.1) \quad s_\mu s_\nu = \sum_{\lambda} c_{\mu, \nu}^{\lambda} s_{\lambda},$$

and the $c_{\mu, \nu}^{\lambda}$ are called **Littlewood–Richardson coefficients**. We will see some special cases soon and study this in more depth later. From the definition, we have

$$c_{\mu, \nu}^{\lambda} = c_{\nu, \mu}^{\lambda}.$$

Applying ω to (2.7.1), we get

$$(2.7.2) \quad c_{\mu, \nu}^{\lambda} = c_{\nu^{\dagger}, \mu^{\dagger}}^{\lambda^{\dagger}}.$$

We can give an interpretation for the Littlewood–Richardson coefficients in the special case where μ (or ν) has a single part or all parts equal to 1. Say that λ/ν is a **horizontal strip** if no column in the skew Young diagram of λ/ν contains 2 or more boxes. Similarly, say that λ/ν is a **vertical strip** if no row in the skew Young diagram of λ/ν contains 2 or more boxes.

Theorem 2.7.3 (Pieri rule). • If $\mu = (1^k)$, then

$$c_{(1^k), \nu}^{\lambda} = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ and } \lambda/\nu \text{ is a vertical strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_{\nu} s_{1^k} = \sum_{\lambda} s_{\lambda}$$

where the sum is over all λ such that λ/ν is a vertical strip of size k .

- If $\mu = (k)$, then

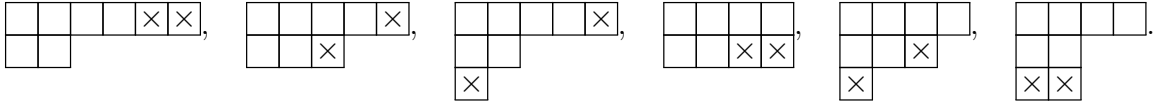
$$c_{(k),\nu}^\lambda = \begin{cases} 1 & \text{if } |\lambda| = |\nu| + k \text{ and } \lambda/\nu \text{ is a horizontal strip} \\ 0 & \text{otherwise} \end{cases}.$$

In other words,

$$s_\nu s_k = \sum_{\lambda} s_\lambda$$

where the sum is over all λ such that λ/ν is a horizontal strip of size k .

Example 2.7.4. To multiply s_λ by s_k , it suffices to enumerate all partitions that we can get by adding k boxes to the Young diagram of λ , no two of which are in the same column. For example, here we have drawn all of the ways to add 2 boxes to $(4, 2)$:



So

$$s_{4,2}s_2 = s_{6,2} + s_{5,3} + s_{5,2,1} + s_{4,4} + s_{4,3,1} + s_{4,2,2}.$$

□

Corollary 2.7.5. $s_\nu e_\mu = \sum_{\lambda} K_{\lambda^\dagger/\nu^\dagger, \mu} s_\lambda$.

Corollary 2.7.6. $s_\nu h_\mu = \sum_{\lambda} K_{\lambda/\nu, \mu} s_\lambda$.

Recall that for $|\lambda| = n$, $f^\lambda = K_{\lambda, 1^n}$ is the number of standard Young tableaux of shape λ .

Corollary 2.7.7. $s_1^n = \sum_{\lambda \vdash n} f^\lambda s_\lambda$.

Proof. The Pieri rule says that to multiply s_1^n , we first enumerate all sequences $\lambda^{(1)} \subset \lambda^{(2)} \subset \dots \subset \lambda^{(n)}$ where $|\lambda^{(i)}| = i$. Then the result is the sum of s_λ with multiplicity given by the number of sequences with $\lambda^{(n)} = \lambda$. But such sequences are in bijection with standard Young tableaux: label the unique box in $\lambda^{(i)}/\lambda^{(i-1)}$ with i . □

Remark 2.7.8. From the interpretation of s_λ as the character of an irreducible representation \mathbf{S}_λ , and the fact that polynomial representations are direct sums of irreducible ones, we can reinterpret the Littlewood–Richardson coefficient as the multiplicity of \mathbf{S}_λ in the decomposition of the tensor product of $\mathbf{S}_\mu \otimes \mathbf{S}_\nu$. From this, it is immediate that $c_{\mu, \nu}^\lambda \geq 0$.

The Pieri rule describes the decomposition of the tensor product of \mathbf{S}_λ with an exterior power \bigwedge^k , respectively, a symmetric power Sym^k . □

2.8. Tensor categories.

2.8.1. *Definitions.* We will make use of tensor categories. We won't really need the precise definition, but we'll go over the main points. Let \mathcal{A} be an abelian category. Roughly, this means that the objects can be thought of as the category of modules over some ring. In particular, the category of modules over a fixed ring is an example. Practically, it means that notions such as kernels, cokernels, exactness, etc. can be defined, and analogues of the isomorphism theorems for modules hold. Also, the morphisms between two fixed objects form an abelian group.

Even though we haven't made the definition precise, here is one easy construction that we will make a lot of use of. First, given two categories \mathcal{C} and \mathcal{A} , the functor category $\text{Fun}(\mathcal{C}, \mathcal{A})$ is the category whose objects are functors $\mathcal{C} \rightarrow \mathcal{A}$ and whose morphisms are natural

transformations. If \mathcal{A} is abelian, then so is $\text{Fun}(\mathcal{C}, \mathcal{A})$: things such as kernels, cokernels, etc. are computed pointwise, meaning that $\ker(F \rightarrow G)(X) = \ker(F(X) \rightarrow G(X))$, etc. We'll usually apply this when \mathcal{A} is the category of modules over a ring, or usually just the category of vector spaces over a field.

A **monoidal structure**, or tensor product, on \mathcal{A} is a functor $\otimes: \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, written $X \otimes Y$ rather than $\otimes(X, Y)$, which satisfies associativity in the sense that we are given an isomorphism $(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$ which satisfies some axioms, together with a unit object $1_{\mathcal{A}}$ (or just 1) which satisfies $1 \otimes X \cong X \otimes 1$. Again, we won't make so much use of the fine details of the definition, rather we'll work with a few concrete examples. Given a tensor product, a **symmetry** is a natural isomorphism $\tau_{X,Y}: X \otimes Y \cong Y \otimes X$ given for all objects X, Y such that $\tau_{X,Y} = \tau_{Y,X}^{-1}$ that satisfy the braid relations. To expand on the last point: we can apply the $\tau_{X,X}$ to a 3-fold tensor product $X^{\otimes 3}$ by either working on first two factors (call it $\tau_{1,2}$) or the last two factors (call it $\tau_{2,3}$). The braid relation is:

$$\tau_{1,2}\tau_{2,3}\tau_{1,2} = \tau_{2,3}\tau_{1,2}\tau_{2,3}.$$

This is the relation that the transpositions (1, 2) and (2, 3) in Σ_3 satisfy. In particular, this implies that the symmetric group Σ_n naturally acts on any n -fold tensor product $X^{\otimes n}$.

Example 2.8.1. The usual tensor product gives a monoidal structure on the category of modules over a commutative ring. \square

Given two tensor categories \mathcal{A}, \mathcal{B} , a **tensor functor** is a functor $F: \mathcal{A} \rightarrow \mathcal{B}$ together with natural isomorphisms $F(X \otimes Y) \cong F(X) \otimes F(Y)$ and $F(1_{\mathcal{A}}) \cong 1_{\mathcal{B}}$ satisfying some axioms. Again, we will not go into the details.

An important feature is that we can redo a lot of multilinear algebra in the setting of a symmetric tensor category. For example, the symmetric square of an object $S^2(X)$ is the cokernel of the map

$$\tau_{X,X} - 1_{X \otimes X}: X \otimes X \rightarrow X \otimes X,$$

and similarly one can define higher symmetric powers by taking the quotient of these maps for all adjacent positions. We define the exterior power $\bigwedge^n(X)$ as the image of the map

$$\sum_{\sigma \in \Sigma_n} \text{sgn}(\sigma)\sigma.$$

The multiplication maps $S^n(X) \otimes S^m(X) \rightarrow S^{n+m}(X)$ are defined as well. We can also define the comultiplication maps $\bigwedge^n(X) \rightarrow \bigwedge^i(X) \otimes \bigwedge^{n-i}(X)$ as follows. Pick coset representatives $\sigma_1, \dots, \sigma_N$ for $\Sigma_n/\Sigma_i \times \Sigma_{n-i}$ and consider the map

$$\sum_{i=1}^N \text{sgn}(\sigma_i)\sigma_i$$

on $X^{\otimes n}$. This maps $\bigwedge^n(X)$ into $\bigwedge^i(X) \otimes \bigwedge^{n-i}(X)$ and is the desired map. Once we have these maps, we can also define arbitrary Schur functors $\mathbf{S}_{\lambda}(X)$ using the definition from the previous section.

2.8.2. Example: chain complexes. Given a commutative ring R , a chain complex V_{\bullet} is a sequence of R -modules and R -linear maps (differentials):

$$\cdots \xrightarrow{d_i} V_i \xrightarrow{d_{i-1}} V_{i-1} \xrightarrow{d_{i-2}} \cdots$$

such that $d_{i-1}d_i = 0$ for all i . A map of chain complexes $f: V_\bullet \rightarrow W_\bullet$ is a sequence of linear maps $f_i: V_i \rightarrow W_i$ such that $f_i d_i^V = d_i^W f_{i+1}$ for all i . The notions of image complex, kernel, cokernel, etc. can be computed term by term. Given chain complexes V_\bullet and W_\bullet , their tensor product is the chain complex with spaces

$$(V \otimes W)_n = \bigoplus_{i+j=n} V_i \otimes W_j$$

and the differentials are the sums of the maps

$$V_i \otimes W_j \rightarrow (V_{i-1} \otimes W_j) \oplus (V_i \otimes W_{j-1})$$

where the map is $d_{i-1}^V \otimes 1 + (-1)^i 1 \otimes d_{j-1}^W$.

The symmetry isomorphism $V_\bullet \otimes W_\bullet \cong W_\bullet \otimes V_\bullet$ is defined as the sum of the isomorphisms

$$\begin{aligned} V_i \otimes W_j &\rightarrow W_j \otimes V_i \\ \sum x \otimes y &\mapsto (-1)^{ij} \sum y \otimes x. \end{aligned}$$

Remark 2.8.2. We can think of the category of R -modules as a subcategory of the category of chain complexes V_\bullet where $V_i = 0$ for $i \neq 0$. Then the symmetry just defined restricts to the usual one. However, we could also think of R -modules as complexes where $V_i = 0$ for $i \neq 1$. Then the symmetry restricts to a twisted version of the usual symmetry which will come up later. \square

2.8.3. Example: symmetric sequences. Let \mathbf{FB} be the category of finite sets and bijections.

Let \mathbf{k} be a field and let $\mathbf{Vec}_{\mathbf{k}}$ the category of \mathbf{k} -vector spaces.

Set $\mathcal{V}_{\mathbf{k}} := \mathbf{Fun}(\mathbf{FB}, \mathbf{Vec}_{\mathbf{k}})$. Given an object $F \in \mathcal{V}_{\mathbf{k}}$ and a finite set S , we get a vector space with an action of $\mathbf{Aut}(S)$, which is isomorphic to a symmetric group. This information is already encoded by just considering the objects of the form $[n]$, but it will be convenient for various constructions to consider all finite sets. So the data of an object of $\mathcal{V}_{\mathbf{k}}$ is the same as a sequence of representations V_n of symmetric groups Σ_n , one for each n .

Given a vector space M , define $M[i] \in \mathcal{V}_{\mathbf{k}}$ by

$$S \mapsto \begin{cases} M & \text{if } |S| = i \\ 0 & \text{if } |S| \neq i \end{cases}$$

and all morphisms act by the identity.

Put a monoidal structure on $\mathcal{V}_{\mathbf{k}}$ as follows. Given $V, W \in \mathcal{V}_{\mathbf{k}}$, we define their tensor product to be

$$(V \otimes W)(S) = \bigoplus_{T \subseteq S} V(T) \otimes_{\mathbf{k}} W(S \setminus T).$$

This has a symmetry τ given by interchanging factors. The unit is $\mathbf{k}[0]$.

Remark 2.8.3. This is one instance where it can be convenient to use all finite sets, not just those of the form $[n]$. However, we can also phrase things from this perspective: given sequences (V_n) and (W_n) , their tensor product is the sequence defined by

$$(V \otimes W)_n = \bigoplus_{i=0}^n \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (V_i \otimes W_{n-i})$$

where for finite groups $H \subset G$, and an H -representation V , the induction is defined by $\text{Ind}_H^G V = \mathbf{k}[G] \otimes_{\mathbf{k}[H]} V$ and $\Sigma_i \times \Sigma_{n-i}$ is a subgroup of Σ_n that only permutes $1, \dots, i$ and $i+1, \dots, n$ separately. \square

So we can define tensor powers $V^{\otimes n}$, and τ allows one to define symmetric powers $\text{Sym}^n(V)$ and exterior powers $\bigwedge^n(V)$ as quotients of $V^{\otimes n}$.

Example 2.8.4.

$$U[1]^{\otimes n}(S) = \begin{cases} \bigoplus_{\sigma \in \text{Aut}(S)} U^{\otimes S} \cong (U^{\otimes n})^{\oplus n!} & \text{if } |S| = n \\ 0 & \text{else} \end{cases}$$

$$\text{Sym}^n(U[1])(S) = \begin{cases} U^{\otimes S} & \text{if } |S| = n \\ 0 & \text{else} \end{cases}$$

The action of $\text{Aut}(S)$ on $U^{\otimes S}$ is by permuting tensor factors. \square

$\mathcal{V}_{\mathbf{k}}$ has a subcategory $\mathcal{V}_{\mathbf{k}}^f$ of “locally finite” objects: those functors such that $\dim V(S) < \infty$ for all S .

2.8.4. *Example: polynomial functors.* Now assume that \mathbf{k} is an infinite field. Write $\mathbf{Vec}_{\mathbf{k}}$ for the category of finite dimensional vector spaces over \mathbf{k} .

Definition 2.8.5. A functor $F: \mathbf{Vec}_{\mathbf{k}} \rightarrow \mathbf{Vec}_{\mathbf{k}}$ is **polynomial** if for all V, V' , the map

$$\text{Hom}_{\mathbf{Vec}_{\mathbf{k}}}(V, V') \rightarrow \text{Hom}_{\mathbf{Vec}_{\mathbf{k}}}(F(V), F(V'))$$

is defined by polynomial functions. If these polynomials are homogeneous of degree d , we say that F is a degree d polynomial functor. Let $\mathbf{Pol}_{\mathbf{k}}$ be the category of polynomial functors, and let $\mathbf{Pol}_{\mathbf{k},d}$ be the subcategory of degree d polynomial functors. \square

Note that a polynomial functor is naturally a direct sum of its homogeneous parts, so we have $\mathbf{Pol}_{\mathbf{k}} = \bigoplus_{d \geq 0} \mathbf{Pol}_{\mathbf{k},d}$.

The Schur functors \mathbf{S}_{λ} are examples of polynomial functors. They are homogeneous of degree $|\lambda|$.

Theorem 2.8.6. *If \mathbf{k} is a field of characteristic 0, then every irreducible polynomial functor is a Schur functor.*

$\mathbf{Pol}_{\mathbf{k}}$ has a tensor product:

$$(F \otimes F')(V) = F(V) \otimes_{\mathbf{k}} F'(V)$$

and a symmetry τ which interchanges the factors.

$\mathbf{Pol}_{\mathbf{k}}$ has a subcategory $\mathbf{Pol}_{\mathbf{k}}^f$ of locally finite objects: those functors such that the degree d piece is a finite length functor for all d , i.e., there are only finitely many subfunctors.

2.8.5. *Algebras and modules.* Let (\mathcal{A}, \otimes) be a monoidal category with unit 1. An **algebra** is an object A together with maps

$$\mu: A \otimes A \rightarrow A, \quad e: 1 \rightarrow A$$

which should satisfy the axioms of being an associative, unital algebra (thinking of μ as multiplication and e as inclusion of the unit). More precisely, it means that the following

diagram should commute:

$$\begin{array}{ccc} A \otimes A \otimes A & \xrightarrow{\mu \otimes \text{id}} & A \otimes A \\ \text{id} \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes A & \xrightarrow{\mu} & A \end{array}$$

and $\mu \circ (e \otimes \text{id}): 1 \otimes A \rightarrow A \otimes A \rightarrow A$ is the same as the natural isomorphism $1 \otimes A \cong A$ and similarly with $\mu \circ (\text{id} \otimes e)$.

Given an algebra A , a (left) A -module is an object M together with a map $\mu: A \otimes M \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccc} A \otimes A \otimes M & \xrightarrow{\mu \otimes \text{id}} & A \otimes M \\ \text{id} \otimes \mu \downarrow & & \downarrow \mu \\ A \otimes M & \xrightarrow{\mu} & M \end{array}$$

and axioms similar to the unit axioms for A are satisfied.

If \mathcal{A} is also equipped with a symmetry τ , then we can also define the notion of a commutative algebra. This is an algebra as above such that $\mu = \mu \circ \tau$ as maps $A \otimes A \rightarrow A$. The definition of a module over a commutative algebra remains unchanged.

2.9. Categorical version of Schur–Weyl duality. The decomposition of s_1^n can be interpreted as a decomposition of the tensor power of a vector space

$$(\mathbf{C}^d)^{\otimes n} = \bigoplus_{\substack{\lambda \vdash n \\ \ell(\lambda) \leq d}} \mathbf{S}_\lambda(\mathbf{C}^d)^{\oplus f^\lambda}.$$

Hence the multiplicity space of $\mathbf{S}_\lambda(\mathbf{C}^d)$ has dimension f^λ . By Schur–Weyl duality, we see that f^λ is the dimension of an irreducible representation of Σ_n , which we will call \mathbf{M}_λ . In the next section, we will discuss a construction for \mathbf{M}_λ . For now, one can take the multiplicity space of $\mathbf{S}_\lambda \mathbf{C}^d$ in $(\mathbf{C}^d)^{\otimes n}$.

Hence, the indexing set in Schur–Weyl duality can be taken to be the set of partitions of size n with at most d parts. If $d \geq n$, this latter condition is unnecessary.

Define a functor $\Phi: \text{Fun}(\mathbf{FB}, \mathbf{Vec}_{\mathbf{k}}) \rightarrow \mathbf{Pol}_{\mathbf{k}}$ by $V \mapsto \Phi_V$ where Φ_V is defined by:

$$\Phi_V(W) = \bigoplus_{n \geq 0} (V_n \otimes_{\mathbf{k}} W^{\otimes n})^{\Sigma_n},$$

and we write V_i for $V([i])$. Here Σ_n acts on $W^{\otimes n}$ by permuting factors, and the superscript denotes invariants.

Proposition 2.9.1. Φ is a tensor functor between $\mathcal{V}_{\mathbf{k}}^f$ and $\mathbf{Pol}_{\mathbf{k}}^f$.

If \mathbf{k} has characteristic 0, then Φ is an equivalence between $\mathcal{V}_{\mathbf{k}}^f$ and $\mathbf{Pol}_{\mathbf{k}}^f$.

Proof. Pick $V, V' \in \text{Fun}(\mathbf{FB}, \mathbf{Vec}_k)$. Then we have

$$\begin{aligned}
 (\Phi_V \otimes \Phi_{V'})(W) &= \Phi_V(W) \otimes \Phi_{V'}(W) \\
 &= \left(\bigoplus_{n \geq 0} (V_n \otimes W^{\otimes n})^{\Sigma_n} \right) \otimes \left(\bigoplus_{n \geq 0} (V'_n \otimes W^{\otimes n})^{\Sigma_n} \right) \\
 &= \bigoplus_{n \geq 0} \bigoplus_{i=0}^n (V_i \otimes W^{\otimes i})^{\Sigma_i} \otimes (V'_{n-i} \otimes W^{\otimes(n-i)})^{\Sigma_{n-i}} \\
 &= \bigoplus_{n \geq 0} \bigoplus_{i=0}^n (\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (V_i \otimes V'_{n-i}) \otimes W^{\otimes n})^{\Sigma_n} \\
 &= \Phi_{V \otimes V'}(W).
 \end{aligned}$$

The fourth equality comes from hom-tensor adjunction (Frobenius reciprocity). To elaborate: since everything is finite-dimensional, we can write (below Res denotes the restriction of a representation to a subgroup)

$$\begin{aligned}
 (\text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (V_i \otimes V'_{n-i}) \otimes W^{\otimes n})^{\Sigma_n} &= \text{Hom}_{\Sigma_n}((W^{\otimes n})^*, \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (V_i \otimes V'_{n-i})) \\
 &= \text{Hom}_{\Sigma_i \times \Sigma_{n-i}}(\text{Res}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (W^{\otimes n})^*, V_i \otimes V'_{n-i}) \\
 &= \text{Hom}_{\Sigma_i}((W^{\otimes i})^*, V_i) \otimes \text{Hom}_{\Sigma_{n-i}}((W^{\otimes(n-i)})^*, V'_{n-i}) \\
 &= (V_i \otimes W^{\otimes i})^{\Sigma_i} \otimes (V'_{n-i} \otimes W^{\otimes(n-i)})^{\Sigma_{n-i}}.
 \end{aligned}$$

For the last statement, first note that every locally finite object of \mathcal{V}_k is a direct sum of \mathbf{M}_λ , where we think of \mathbf{M}_λ as a functor that has nonzero values only for sets of size $|\lambda|$ and only finitely many copies are used for each λ . By Schur–Weyl duality, the image of \mathbf{M}_λ is \mathbf{S}_λ . In particular, Φ induces a bijection between the sets of irreducible objects of both categories, and every object is a direct sum of irreducible objects (note that Φ preserves direct sums), so we are done. \square

Remark 2.9.2. For a direct construction of \mathbf{M}_λ , see [F, §7.2]. \square

Corollary 2.9.3. *Given partitions λ, μ of size n and m , we have*

$$\text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} \mathbf{M}_\lambda \boxtimes \mathbf{M}_\mu = \bigoplus_{\nu} \mathbf{M}_\nu^{\oplus c'_{\lambda, \mu}}.$$

Furthermore, for any partition ν of size p , have

$$\text{Res}_{\Sigma_i \times \Sigma_{p-i}}^{\Sigma_p} \mathbf{M}_\nu = \bigoplus_{\lambda, \mu} (\mathbf{M}_\lambda \boxtimes \mathbf{M}_\mu)^{\oplus c'_{\lambda, \mu}}.$$

Proof. The first part comes from the fact that the Littlewood–Richardson coefficients describe multiplication for \mathbf{GL} -representations. The second part follows from the first by Frobenius reciprocity. \square

2.10. Infinite number of variables. There is a third symmetric monoidal category which will be convenient to use. We have inclusions

$$\begin{aligned}
 \mathbf{GL}_n(\mathbf{C}) &\rightarrow \mathbf{GL}_{n+1}(\mathbf{C}) \\
 X &\mapsto \begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix},
 \end{aligned}$$

and we define

$$\mathbf{GL}_\infty(\mathbf{C}) = \bigcup_{n \geq 1} \mathbf{GL}_n(\mathbf{C}).$$

This group consists of infinite invertible matrices which differ from the identity matrix only in finitely many entries. Let T denote the subgroup of diagonal matrices. Say that a polynomial representation of $\mathbf{GL}_\infty(\mathbf{C})$ is **T -semisimple** if its restriction to T decomposes as a direct sum of 1-dimensional representations. These will be polynomial representations of T , which are classified by non-negative integer sequences $(\alpha_1, \alpha_2, \dots)$. Furthermore, since it comes a polynomial representation of $\mathbf{GL}_\infty(\mathbf{C})$, these sequences can only have a finite number of nonzero entries. The representation $\mathbf{C}_\alpha = \mathbf{C}$ indexed by α is described by

$$\text{diag}(x_1, x_2, \dots) \cdot 1 = \prod_{i \geq 1} x_i^{\alpha_i}.$$

We let $\text{Rep}(\mathbf{GL})$ be the category of polynomial T -semisimple representations of $\mathbf{GL}_\infty(\mathbf{C})$ such that each representation \mathbf{C}_α appears with finite multiplicity. Given $V \in \text{Rep}(\mathbf{GL})$, let V_n be the sum of the \mathbf{C}_α such that $\sum_i \alpha_i = n$. Then $V = \bigoplus_{n \geq 0} V_n$ is a decomposition of V into a direct sum of subrepresentations. Say that V is locally finite if each V_n has a finite composition series. We let $\text{Rep}^f(\mathbf{GL})$ be the full subcategory of $\text{Rep}(\mathbf{GL})$ consisting of the locally finite representations. Let V_α be the direct sum of the \mathbf{C}_α that appear in V .

Given an object V of $\text{Rep}(\mathbf{GL})$, let $V(n)$ denote the direct sum of those V_α such that $\alpha_i = 0$ for $i > n$. Then $V(n)$ can also be thought of as the invariant subspace of diagonal matrices whose first n entries are 1. Since the subgroup $\mathbf{GL}_n(\mathbf{C})$ commutes with these diagonal matrices, it acts on $V(n)$. Furthermore, we have $V = \bigcup_{n \geq 1} V(n)$. Note that there is a natural tensor product on objects of $\text{Rep}(\mathbf{GL})$ (the conditions we imposed are closed under tensor products).

Lemma 2.10.1. *The assignment $\mathbf{C}^n \mapsto V(n)$ is a polynomial functor.*

Proof. Since V is a polynomial representation, the function $\rho: \mathbf{GL}_\infty(\mathbf{C}) \rightarrow \mathbf{GL}(V)$ can be extended to $\text{End}(\mathbf{C}^\infty) \rightarrow \text{End}(V)$ (the entries of ρ are polynomials and don't require invertibility of the input matrix). Given any linear map $f: \mathbf{C}^n \rightarrow \mathbf{C}^m$, we can extend this to an element $f \in \text{End}(\mathbf{C}^\infty)$ and this gives a linear map $V(n) \rightarrow V(m)$ given by polynomial functions. Since each \mathbf{C}_α appears with finite multiplicity, $V(n)$ is finite-dimensional. \square

Call the polynomial functor $\mathbf{C}^n \mapsto V(n)$ by F_V .

Theorem 2.10.2. *The assignment $V \mapsto F_V$ defines an equivalence between $\text{Rep}(\mathbf{GL})$ and Pol of symmetric monoidal categories.*

Proof. To define the inverse, let F be a polynomial functor. The inclusion maps $\mathbf{C}^n \rightarrow \mathbf{C}^{n+1}$ give maps $F(\mathbf{C}^n) \rightarrow F(\mathbf{C}^{n+1})$ (also inclusions because the projection $\mathbf{C}^{n+1} \rightarrow \mathbf{C}^n$ induce a left inverse after applying F). Then we get a polynomial representation of $\mathbf{GL}_\infty(\mathbf{C})$ by taking $\bigcup_{n \geq 1} F(\mathbf{C}^n)$. The fact that the symmetric monoidal structures is preserved just comes from the fact that it is defined by a tensor product of vector spaces in both cases. \square

Given $V \in \text{Rep}(\mathbf{GL})$, we can define its character by

$$\text{char}(V) = \sum_{\alpha} \dim(V_\alpha) \prod_{i \geq 1} x_i^{\alpha_i}.$$

Lemma 2.10.3. $\text{char}(V) \in \Lambda$.

Proof. Proof is similar to the case of polynomial representations of $\mathbf{GL}_n(\mathbf{C})$. □

This is one natural context for symmetric functions, and the assignment $V \mapsto V(n)$ on the level of characters corresponds to setting $x_i = 0$ for $i > n$.

Given a polynomial functor F which is a direct sum of Schur functors (i.e., any element in \mathbf{Pol}^f), we define $\ell(F)$ to be the supremum of $\ell(\lambda)$ such that \mathbf{S}_λ appears in F . Note that $\mathbf{S}_\lambda(\mathbf{C}^n) \neq 0$ if and only if $n \geq \ell(\lambda)$, so $\ell(F)$ tells us how big n must be so that $F(\mathbf{C}^n)$ doesn't lose any information (i.e., kill any submodules). An advantage to working with $\text{Rep}(\mathbf{GL})$ is that we are essentially working with the case $n = \infty$, and the equivalence above is stating that we never lose information in this case.

2.11. Littlewood–Richardson rule. We now want to understand the decomposition

$$\text{Ind}_{\Sigma_n \times \Sigma_m}^{\Sigma_{n+m}} (\mathbf{M}_\lambda \otimes \mathbf{M}_\mu) = \bigoplus_{\nu} \mathbf{M}_\nu^{\oplus c_{\lambda,\mu}^\nu},$$

or equivalently, the decomposition

$$\mathbf{S}_\lambda \otimes \mathbf{S}_\mu = \bigoplus_{\nu} \mathbf{S}_\nu^{\oplus c_{\lambda,\mu}^\nu},$$

where the sum is over all partitions ν of $n + m$ and $c_{\lambda,\mu}^\nu$ is a non-negative integer. The numbers $c_{\lambda,\mu}^\nu$ are called the **Littlewood–Richardson coefficients**, and the eponymous rule gives a combinatorial description of them.

There are many descriptions for this rule, and we will formulate it via lattice words. First consider the skew-diagram ν/λ . We fill the boxes with positive integers so that i appears exactly μ_i times. Then $c_{\lambda,\mu}^\nu$ counts the number of such fillings which satisfy the properties (Littlewood–Richardson tableaux):

- semistandard: the entries are weakly increasing from left to right in each row, and the entries are strictly increasing from top to bottom in each column
- lattice word: Read the entries right to left in each row, starting with the top row to get a sequence of positive integers (reading word). Then each initial segment of this sequence has the property that for each i , i occurs at least as many times as $i + 1$.

See [F, §5, §7.3] or [Mac, §I.9]. See also [Sta, Appendix 7.A.1.3] for some other formulations of the rule. Here are some simple consequences of the Littlewood–Richardson rule:

- If $c_{\lambda,\mu}^\nu \neq 0$, then $\lambda \subseteq \nu$ and $\mu \subseteq \nu$.
- For all partitions λ, μ , $c_{\lambda,\mu}^{\lambda+\mu} = 1$, and $c_{\lambda,\mu}^{\lambda \cup \mu} = 1$ where $\lambda \cup \mu$ denotes the partition obtained by sorting the sequence (λ, μ) . To prove these, fill the Young diagram of μ with the number i in each box in the i th row. Append the i th row to the i th row of λ to see $c_{\lambda,\mu}^{\lambda+\mu} \geq 1$. Append the i th column to the i th column of λ to see $c_{\lambda,\mu}^{\lambda \cup \mu} \geq 1$. The reverse inequalities follow by the extremality of these shapes.
- For all integers $N > 0$, we have $c_{N\lambda, N\mu}^{N\nu} \geq c_{\lambda,\mu}^\nu$, which can be seen by “stretching” the Littlewood–Richardson tableau. As a consequence, if $c_{\lambda,\mu}^\nu > 0$, then $c_{N\lambda, N\mu}^{N\nu} > 0$ for any $N > 0$. The converse of this statement is also true, i.e., if $c_{N\lambda, N\mu}^{N\nu} > 0$ for some $N > 0$, then $c_{\lambda,\mu}^\nu > 0$. This is a highly non-trivial fact known as the saturation theorem, see [KT1, DW1, KM] for different proofs of it. Furthermore, the function $N \mapsto C_{N\lambda, N\mu}^{N\nu}$ is a polynomial in $N \geq 0$ for any fixed choice of λ, μ, ν [DW2, Corollary 3].

And here are some properties which are not obvious from the Littlewood–Richardson rule, but follow easily from the representation-theoretic interpretation:

- Symmetry: $c_{\lambda,\mu}^\nu = c_{\mu,\lambda}^\nu$. One way to give a symmetric combinatorial rule for $c_{\lambda,\mu}^\nu$ is to use the plactic monoid and jeu de taquin [F, §2, §5.1].
- Transpose symmetry: $c_{\lambda^\dagger,\mu^\dagger}^{\nu^\dagger} = c_{\lambda,\mu}^\nu$.

Example 2.11.1. We calculate $c_{(3,1),(4,2,1)}^{(5,3,2,1)} = 3$. The Littlewood–Richardson tableaux are

			1	1
	1	1		
2	2			
3				

			1	1
	1	2		
1	2			
3				

			1	1
	1	2		
1	3			
2				

The reading words are 1111223, 1121213, and 1121312, respectively. It is easier to calculate this number after swapping the roles of $(3, 1)$ and $(4, 2, 1)$:

				1
		2		
	1			
1				

				1
		1		
	2			
1				

				1
		1		
	1			
2				

□

2.12. A few more formulas. By the equivalences above, the induction product on symmetric group representations corresponds to the tensor product of polynomial functors, or tensor product of representations in $\text{Rep}(\mathbf{GL})$.

Proposition 2.12.1. *We have a $\mathbf{GL}_n \times \mathbf{GL}_m$ -equivariant decomposition*

$$\mathbf{S}_\nu(\mathbf{C}^n \oplus \mathbf{C}^m) = \bigoplus_{\lambda,\mu} (\mathbf{S}_\lambda(\mathbf{C}^n) \boxtimes \mathbf{S}_\mu(\mathbf{C}^m))^{\oplus c_{\lambda,\mu}^\nu}.$$

Proof. Set $k = |\nu|$. We have

$$\mathbf{S}_\nu(\mathbf{C}^n \oplus \mathbf{C}^m) = \text{Hom}_{\Sigma_k}(\mathbf{M}_\nu, (\mathbf{C}^{n+m})^{\otimes k}).$$

Now $(\mathbf{C}^{n+m})^{\otimes k}$ can be written as $\bigoplus_I \bigotimes_{j=1}^k A_{j,I}$ where the sum is over all subsets of $\{1, \dots, k\}$ and $A_{j,I} = \mathbf{C}^n$ if $j \in I$ and $A_{j,I} = \mathbf{C}^m$ if $j \notin I$. For every N , the symmetric group Σ_k preserves the sum $\bigoplus_{|I|=N} \bigotimes_j A_{j,I}$, and this representation is the induced representation

$$\text{Ind}_{\Sigma_N \times \Sigma_{k-N}}^{\Sigma_k} ((\mathbf{C}^n)^{\otimes N} \otimes (\mathbf{C}^m)^{\otimes (k-N)}).$$

Hence by Frobenius reciprocity, we can write

$$\begin{aligned} \mathbf{S}_\nu(\mathbf{C}^n \oplus \mathbf{C}^m) &= \bigoplus_{N=0}^k \text{Hom}_{\Sigma_N \times \Sigma_{k-N}}(\mathbf{M}_\nu|_{\Sigma_N \times \Sigma_{k-N}}, (\mathbf{C}^n)^{\otimes N} \otimes (\mathbf{C}^m)^{\otimes (k-N)}) \\ &= \bigoplus_{N=0}^k \bigoplus_{\lambda,\mu} \text{Hom}_{\Sigma_N \times \Sigma_{k-N}}(\mathbf{M}_\lambda \otimes \mathbf{M}_\mu, (\mathbf{C}^n)^{\otimes N} \otimes (\mathbf{C}^m)^{\otimes (k-N)})^{\oplus c_{\lambda,\mu}^\nu} \\ &= \bigoplus_{\lambda,\mu} (\mathbf{S}_\lambda(\mathbf{C}^n) \boxtimes \mathbf{S}_\mu(\mathbf{C}^m))^{\oplus c_{\lambda,\mu}^\nu}. \end{aligned}$$

This gives a $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ -equivariant decomposition of $\mathbf{S}_\nu(\mathbf{C}^n \oplus \mathbf{C}^m)$, and we see that the multiplicities are described by Littlewood–Richardson coefficients. □

The induction product of symmetric group representations is natural from the perspective of our equivalence with polynomial functors, but one can also consider tensor products of symmetric group representations. Again, we will have certain decompositions

$$\mathbf{M}_\lambda \otimes \mathbf{M}_\mu = \bigoplus_{\nu} \mathbf{M}_\nu^{g_{\lambda,\mu}^\nu}.$$

In fact, the \mathbf{M}_λ are isomorphic to their own duals since they can be defined over the rational numbers (it is enough to be defined over the real numbers), so we have

$$g_{\lambda,\mu}^\nu = \dim_{\mathbf{C}}(\mathbf{M}_\lambda \otimes \mathbf{M}_\mu \otimes \mathbf{M}_\nu)^{\Sigma_n}$$

from which it is clear that the coefficient is symmetric in λ, μ, ν , so we denote them by $g_{\lambda,\mu,\nu}$. These are the **Kronecker coefficients** and it is a notoriously difficult problem to give subtraction-free combinatorial rules for them. A special case is when $\lambda = (n)$, since then \mathbf{M}_λ is the trivial representation, in which case we get

$$g_{(n),\mu,\nu} = \delta_{\mu,\nu}.$$

Another easy case is when $\lambda = (1^n)$ since the $\mathbf{M}_{(1^n)}$ is the sign representation. It turns out that $\mathbf{M}_\lambda \otimes \text{sgn} = \mathbf{M}_{\lambda^\dagger}$, and hence

$$g_{(1^n),\mu,\nu} = \delta_{\mu,\nu^\dagger}.$$

Proposition 2.12.2. *We have a $\mathbf{GL}_n \times \mathbf{GL}_m$ -equivariant decomposition*

$$\mathbf{S}_\nu(\mathbf{C}^n \otimes \mathbf{C}^m) = \bigoplus_{\lambda,\mu} (\mathbf{S}_\lambda(\mathbf{C}^n) \boxtimes \mathbf{S}_\mu(\mathbf{C}^m))^{\oplus g_{\lambda,\mu,\nu}}.$$

Proof. First, we have

$$(\mathbf{C}^n \otimes \mathbf{C}^m)^{\otimes k} = \bigoplus_{|\nu|=k} \mathbf{S}_\nu(\mathbf{C}^n \otimes \mathbf{C}^m) \boxtimes \mathbf{M}_\nu$$

as representations of $\mathbf{GL}_{nm}(\mathbf{C}) \times \Sigma_k$. Alternatively, as $\mathbf{GL}(n) \times \mathbf{GL}(m) \times \Sigma_k$ representations, we have

$$\begin{aligned} (\mathbf{C}^n)^{\otimes k} \otimes (\mathbf{C}^m)^{\otimes k} &= \left(\bigoplus_{|\lambda|=k} \mathbf{S}_\lambda(\mathbf{C}^n) \boxtimes \mathbf{M}_\lambda \right) \otimes \left(\bigoplus_{|\mu|=k} \mathbf{S}_\mu(\mathbf{C}^m) \boxtimes \mathbf{M}_\mu \right) \\ &= \bigoplus_{|\lambda|=|\mu|=|\nu|=k} (\mathbf{S}_\lambda(\mathbf{C}^n) \boxtimes \mathbf{S}_\mu(\mathbf{C}^m) \boxtimes \mathbf{M}_\nu)^{\oplus g_{\lambda,\mu,\nu}}. \end{aligned}$$

The result follows by taking the \mathbf{M}_ν -isotypic component of both expressions. \square

Corollary 2.12.3 (Cauchy identities). *For each k , we have $\mathbf{GL}_n(\mathbf{C}) \times \mathbf{GL}_m(\mathbf{C})$ -equivariant decompositions*

$$\begin{aligned} \text{Sym}^k(\mathbf{C}^n \otimes \mathbf{C}^m) &= \bigoplus_{|\lambda|=k} \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_\lambda(\mathbf{C}^m), \\ \bigwedge^k(\mathbf{C}^n \otimes \mathbf{C}^m) &= \bigoplus_{|\lambda|=k} \mathbf{S}_\lambda(\mathbf{C}^n) \otimes \mathbf{S}_{\lambda^\dagger}(\mathbf{C}^m). \end{aligned}$$

Of course, when n, m are small relative to k , we don't need to sum over all partitions: recall that $\mathbf{S}_\lambda(\mathbf{C}^n) = 0$ if $n < \ell(\lambda)$.

3. REPRESENTATIONS OF COMBINATORIAL CATEGORIES

3.1. Twisted commutative algebras. Let \mathbf{k} be a commutative ring. A **twisted commutative algebra (tca)** is a commutative algebra in the tensor category of symmetric sequences over \mathbf{k} . Let's unpack this definition.

Let $A = \bigoplus_{n \geq 0} A_n$ be a $\mathbf{Z}_{\geq 0}$ -graded, associative, unital \mathbf{k} -algebra such that each A_n has a linear action of the symmetric group Σ_n . Embed $\Sigma_n \times \Sigma_m$ as a subgroup of Σ_{n+m} by having Σ_n act on $1, \dots, n$ and Σ_m act on $n+1, \dots, n+m$ in the natural way. Then A is a **twisted commutative algebra (tca)** if, for all n and m ,

- the multiplication map

$$A_n \otimes A_m \rightarrow A_{n+m}$$

is $\Sigma_n \times \Sigma_m$ -equivariant with respect to the embedding just specified, and

- $\tau(xy) = yx$ where $x \in A_n$, $y \in A_m$, and $\tau \in \Sigma_{n+m}$ is the permutation that swaps $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$ in order, i.e., $\tau(i) = m+i$ if $1 \leq i \leq n$ and $\tau(n+j) = j$ for $1 \leq j \leq m$.

Example 3.1.1. Let E be a free \mathbf{k} -module, and $A_n = E^{\otimes n}$ (by convention, $A_0 = \mathbf{k}$). The action of Σ_n is by permuting tensor factors, and multiplication is concatenation of tensors. This is the free tca (i.e., a symmetric algebra in the category of symmetric sequences) generated in degree 1 by E , in the sense that $A_n = \text{Sym}^n(A_1)$ for all n and $A_0 = \mathbf{k}$. We will denote it by $\text{Sym}(E\langle 1 \rangle)$.

When using all sets, we have $A(S) = E^{\otimes S}$, where we can think of $E^{\otimes S}$ as the space of tensors indexed by the set S (more precisely, this should be the colimit of $E^{\otimes n}$ over all bijections $S \cong [n]$, where $n = |S|$). \square

We say that A is **finitely generated** if it is a quotient of $\text{Sym}(V)$ for some finite length symmetric sequence V .

We can also define modules over tca's. In fact, we have a general definition using the monoidal language. We can unpack it as follows. Let A be a tca. An A -module M is a graded abelian group $M = \bigoplus_{n \geq 0} M_n$ such that each M_n is a linear Σ_n -representation, M is a graded A -module in the usual sense, and such that the multiplication map

$$A_n \otimes M_m \rightarrow M_{n+m}$$

is $\Sigma_n \times \Sigma_m$ -equivariant. A module is **finitely generated** if it can be generated by finitely many elements m_1, \dots, m_n under the action of A and the symmetric groups. We are interested in the category of A -modules, and specifically, in formal properties of finitely generated A -modules.

In the language of symmetric sequences, a module M over a tca A is finitely generated if it is a quotient of $A \otimes V$ for some finite length symmetric sequence V (i.e., $\sum \dim V_n < \infty$).

An important notion for us is noetherianity. A module is **noetherian** if all of its submodules are finitely generated. A tca is **noetherian** if all of its finitely generated modules are noetherian. The next facts follow from some standard algebra, and we leave it to the reader:

Proposition 3.1.2. *Let $\mathbf{k}[\Sigma_n]$ be the symmetric sequence which is the regular representation of Σ_n (i.e., Σ_n acting on its group algebra by multiplication on the left) in degree n and 0 elsewhere. A tca A is noetherian if and only if $A \otimes \mathbf{k}[\Sigma_n]$ is noetherian for all n .*

Proposition 3.1.3. *Let M be a module over a tca. The following are equivalent:*

- (1) M is noetherian.

- (2) Submodules of M satisfy the ascending chain condition, i.e., given a chain of submodules $N_1 \subseteq N_2 \subseteq \dots$ of M , we must have $M_i = M_{i+1} = \dots$ for some i .
- (3) Every collection of submodules of M has a maximal element with respect to inclusion.

3.2. Alternative models for tca's generated in degree 1. The tca $\text{Sym}(E\langle 1 \rangle)$ and its modules are an important example, especially when $\dim E = 1$. We can give a different model for $\text{Sym}(E\langle 1 \rangle)$ -modules in terms of functor categories. To do this, pick a basis e_1, \dots, e_d for E .

Let \mathbf{FI}_d be the category such that:

- the objects are finite sets S ,
- a morphism $S \rightarrow T$ is a pair (f, g) where $f: S \rightarrow T$ is an injective function and $g: T \setminus f(S) \rightarrow [d]$ is an arbitrary function (“coloring”).

Given two morphisms $(f, g): S \rightarrow T$ and $(f', g'): T \rightarrow U$, we define the composition to be $(f'', g''): S \rightarrow U$ where $f'' = f' \circ f$, and $g'': U \setminus f''(S) \rightarrow [d]$ is given by $g''(u) = g'(u)$ if u is not in the image of f' , and $g''(u) = g'(v)$ if $u = f'(v)$ and v is not in the image of f .

Proposition 3.2.1. *The category $\text{Fun}(\mathbf{FI}_d, \text{Mod}_{\mathbf{k}})$ is equivalent to the category of $\text{Sym}(E\langle 1 \rangle)$ -modules.*

Proof. Set $A = \text{Sym}(E\langle 1 \rangle)$. Let M be an A -module. Define a functor $F_M: \mathbf{FI}_d \rightarrow \text{Mod}_{\mathbf{k}}$ on objects by setting $F_M(S) = M(S)$ for each finite set S . Consider the multiplication map $A \otimes M \rightarrow M$ evaluated on a set T (indexed slightly differently):

$$\bigoplus_{U \subseteq T} E^{\otimes(T \setminus U)} \otimes M(U) \rightarrow M(T).$$

Given an \mathbf{FI}_d -morphism $(f, g): S \rightarrow T$, we consider the map $E^{\otimes(T \setminus U)} \otimes M(U) \rightarrow M(T)$ above with $U = f(S)$. The coloring $g: T \setminus U \rightarrow [d]$ gives a tensor product x of basis elements in $E^{\otimes(T \setminus U)}$, and hence we get a map $M(S) \rightarrow M(T)$ by considering the restriction to $x \otimes M(U) \rightarrow M(T)$ and identifying U with S via f .

Conversely, given a functor $F: \mathbf{FI}_d \rightarrow \text{Mod}_{\mathbf{k}}$, we define an A -module M_F on objects by $M_F(S) = F(S)$. To define the multiplication map, we need to define the maps $E^{\otimes(T \setminus U)} \otimes M(U) \rightarrow M(T)$ as above for every $U \subseteq T$ and every tensor product x of basis elements. Let $f: U \rightarrow T$ be the inclusion map and let $g: T \setminus U \rightarrow [d]$ be the map corresponding to x . Then we get a map $F(U) \rightarrow F(T)$ which we use to define the desired map.

The two constructions are inverse to each other, so we get the desired equivalence. \square

The special case $d = 1$ will be studied in more depth later. In that case, we write \mathbf{FI} instead of \mathbf{FI}_1 . This is an abbreviation for **F**inite **I**njections.

It will also be useful to have descriptions of $\text{Sym}(E\langle 1 \rangle)$ and its modules in the polynomial functor and \mathbf{GL}_{∞} -models. In that case, we assume that our field is of characteristic 0. Chasing through the equivalence Φ from §2.9, the polynomial functor corresponding to $\text{Sym}(E\langle 1 \rangle)$ in degree n is

$$W \mapsto (E^{\otimes n} \otimes W^{\otimes n})^{\Sigma_n} \cong \text{Sym}^n(E \otimes W),$$

so we get the functor which assigns W to the symmetric algebra on $E \otimes W$. Given a linear map $W \rightarrow W'$, we get an induced linear map $E \otimes W \rightarrow E \otimes W'$ and hence a ring homomorphism $\text{Sym}(E \otimes W) \rightarrow \text{Sym}(E \otimes W')$. Recall the Cauchy identity (Corollary 2.12.3) that

$$\text{Sym}(E \otimes W) = \bigoplus_{\lambda} \mathbf{S}_{\lambda} E \otimes \mathbf{S}_{\lambda} W.$$

This is a $\mathbf{GL}(E) \times \mathbf{GL}(W)$ -equivariant decomposition, and hence gives a decomposition into polynomial functors if we ignore the $\mathbf{GL}(E)$ -action. Note that the action of $\mathbf{GL}(E)$ is not present in any of the definitions for $\mathrm{Sym}(E\langle 1 \rangle)$ -modules. In this language, Proposition 3.1.2 translates to saying that the module $W \mapsto \mathrm{Sym}(E \otimes W) \otimes W^{\otimes n}$ is noetherian for all n .

In the model of representations of $\mathbf{GL}_\infty(\mathbf{C})$ (see §2.10), $\mathrm{Sym}(E\langle 1 \rangle)$ is the algebra

$$\mathrm{Sym}(E \otimes \mathbf{C}^\infty)$$

with the action of $\mathbf{GL}_\infty(\mathbf{C})$ which acts on \mathbf{C}^∞ in the usual way. The latter is a very useful model to keep in mind. A module over this tca is then a $\mathbf{GL}_\infty(\mathbf{C})$ representation (satisfying the conditions in §2.10 with a compatible action of $\mathrm{Sym}(E \otimes \mathbf{C}^\infty)$). In this case, Proposition 3.1.2 translates to saying that the module $\mathrm{Sym}(E \otimes \mathbf{C}^\infty) \otimes (\mathbf{C}^\infty)^n$ is noetherian for all n .

3.3. Bounded tca's. In this section, we continue to work over a field of characteristic 0.

Given a sum of Schur functors $F = \bigoplus_\lambda \mathbf{S}_\lambda^{\oplus c_\lambda}$, we defined $\ell(F)$ to be the supremum of $\ell(\lambda)$ such that $c_\lambda \neq 0$. We say that F is **bounded** if $\ell(F) < \infty$. Given a tca A in the polynomial functor model, we say that A is bounded if it is bounded as a polynomial functor.

Given any vector space \mathbf{C}^n , we can evaluate a tca A (and an A -module M) to get an algebra $A(\mathbf{C}^n)$ with a $\mathbf{GL}_n(\mathbf{C})$ -action together with a module $M(\mathbf{C}^n)$ with a compatible action. This gives a function from the lattice of submodules of M to the lattice of submodules of $M(\mathbf{C}^n)$ which sends $N \subset M$ to $N(\mathbf{C}^n) \subset M(\mathbf{C}^n)$.

Proposition 3.3.1. *Let A be a finitely generated tca. If M is a bounded A -module and $n \geq \ell(M)$, then the map of lattices above is injective. In particular, M is noetherian.*

Proof. Recall that $N(\mathbf{C}^n) = 0$ if and only if $n < \ell(N)$. But $\ell(N) \leq \ell(M)$, so $N(\mathbf{C}^n) \neq 0$ as long as $N \neq 0$. This proves the injectivity statement. For the noetherianity statement, write A as a quotient of $\mathrm{Sym}(V)$ with V a finite length symmetric sequence. Then $A(\mathbf{C}^n)$ is generated as an algebra (in the usual sense) by $V(\mathbf{C}^n)$, a finite dimensional vector space, and $M(\mathbf{C}^n)$ is a finitely generated $A(\mathbf{C}^n)$ in the usual sense. Hence $A(\mathbf{C}^n)$ is a noetherian algebra, and the submodules of $M(\mathbf{C}^n)$ satisfy the ascending chain condition. \square

Proposition 3.3.2. *If A is a bounded tca, then every finitely generated A -module is bounded. In particular, A is noetherian.*

Proof. Let M be a finitely generated A -module and write M as a quotient of $A \otimes V$ with V a finite length symmetric sequence. By §2.11, $\ell(A \otimes V) = \ell(A) + \ell(V) < \infty$. Furthermore, $\ell(M) \leq \ell(A \otimes V)$, so M is bounded. \square

Proposition 3.3.3. *The tca $\mathrm{Sym}(E\langle 1 \rangle)$ is bounded. In particular, it is noetherian.*

Proof. It follows immediately from the Cauchy identity that $\ell(\mathrm{Sym}(E\langle 1 \rangle)) = \dim E$. \square

3.4. Noetherianity in general. Via our equivalence of definitions, we see that $\mathrm{Fun}(\mathbf{FI}_d, \mathrm{Mod}_{\mathbf{k}})$ satisfies some noetherian property when \mathbf{k} is a field of characteristic 0. One can ask whether this holds for general fields. The answer is yes, and we discuss this now and consider when \mathbf{FI}_d is replaced by a more general category. This follows the strategy in [SS1].

We now shift our language slightly. Let \mathcal{C} be a category.³ A **representation** of \mathcal{C} (or a **\mathcal{C} -module**) over \mathbf{k} is a functor $\mathcal{C} \rightarrow \text{Mod}_{\mathbf{k}}$. A map of \mathcal{C} -modules is a natural transformation. We write $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ for the category of representations, which is abelian. Let M be a representation of \mathcal{C} . A subrepresentation is a functor N such that $N(x)$ is a subspace of $M(x)$ for all x and which is closed under all operations $M(f)$ for all morphisms f . By an **element** of M we mean an element of $M(x)$ for some object x of \mathcal{C} . Given any set S of elements of M , there is a smallest subrepresentation of M containing S ; we call this the subrepresentation **generated by** S . We say that M is **finitely generated** if it is generated by a finite set of elements. For a morphism $f: x \rightarrow y$ in \mathcal{C} , we typically write f_* for the given map of \mathbf{k} -modules $M(x) \rightarrow M(y)$.

Let x be an object of \mathcal{C} . Define a representation P_x of \mathcal{C} by $P_x(y) = \mathbf{k}[\text{Hom}_{\mathcal{C}}(x, y)]$, i.e., $P_x(y)$ is the free left \mathbf{k} -module with basis $\text{Hom}_{\mathcal{C}}(x, y)$, the set of morphisms $x \rightarrow y$. For a morphism $f: x \rightarrow y$, we write e_f for the corresponding element of $P_x(y)$. If M is another representation then $\text{Hom}(P_x, M) = M(x)$. This shows that $\text{Hom}(P_x, -)$ is an exact functor, and so P_x is a projective object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$. We call it the **principal projective** at x . A \mathcal{C} -module is finitely generated if and only if it is a quotient of a finite direct sum of principal projective objects.

An object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is **noetherian** if every ascending chain of subobjects stabilizes; this is equivalent to every subrepresentation being finitely generated. The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is **locally noetherian** if every finitely generated object in it is.

Proposition 3.4.1. *The category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is locally noetherian if and only if every principal projective is noetherian.*

Proof. By definition, if $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is locally noetherian then so is every principal projective. Conversely, suppose every principal projective is noetherian. Let M be a finitely generated object. Then M is a quotient of a finite direct sum P of principal projectives. Since noetherianity is preserved under finite direct sums, P is noetherian. And since noetherianity descends to quotients, M is noetherian. This completes the proof. \square

Let $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ be a functor. There is then a pullback functor on representations $\Phi^*: \text{Rep}_{\mathbf{k}}(\mathcal{C}') \rightarrow \text{Rep}_{\mathbf{k}}(\mathcal{C})$ given by $M \mapsto M \circ \Phi$. We study how Φ^* interacts with finiteness conditions. The following definition is of central importance:

Definition 3.4.2. We say that Φ satisfies **property (F)** (F for finite) if the following condition holds: given any object x of \mathcal{C}' there exist finitely many objects y_1, \dots, y_n of \mathcal{C} and morphisms $f_i: x \rightarrow \Phi(y_i)$ in \mathcal{C}' such that for any object y of \mathcal{C} and any morphism $f: x \rightarrow \Phi(y)$ in \mathcal{C}' , there exists a morphism $g: y_i \rightarrow y$ in \mathcal{C} such that $f = \Phi(g) \circ f_i$. \square

The following proposition is the motivation for introducing property (F).

Proposition 3.4.3. *A functor $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ satisfies property (F) if and only if Φ^* takes finitely generated objects of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ to finitely generated objects of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Proof. Assume that Φ satisfies property (F). It suffices to show that Φ^* takes principal projectives to finitely generated representations. Thus let P_x be the principal projective of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ at an object x . Note that $\Phi^*(P_x)(y)$ has for a basis the elements e_f for $f \in \text{Hom}_{\mathcal{C}'}(x, \Phi(y))$. Let $f_i: x \rightarrow \Phi(y_i)$ be as in the definition of property (F). Then the e_{f_i} generate $\Phi^*(P_x)$. The converse is left to the reader (and not used in this paper). \square

³We assume \mathcal{C} is essentially small, that is, the isomorphism classes of objects in \mathcal{C} form a set. In almost all cases, this set is the natural numbers or some slight variant so it won't be an issue for us.

Proposition 3.4.4. *Suppose that $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ is an essentially surjective functor. Let M be an object of $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ such that $\Phi^*(M)$ is finitely generated (resp. noetherian). Then M is finitely generated (resp. noetherian).*

Proof. Let S be a set of elements of $\Phi^*(M)$. Let S' be the corresponding set of elements of M . (Thus if S contains $m \in \Phi^*(M)(y)$ then S' contains $m \in M(\Phi(y))$.) If N is a subrepresentation of M containing S' then $\Phi^*(N)$ is a subrepresentation of $\Phi^*(M)$ containing S . It follows that if N (resp. N') is the subrepresentation of M (resp. $\Phi^*(M)$) generated by S' (resp. S), then $N' \subset \Phi^*(N)$. Thus if S generates $\Phi^*(M)$ then $\Phi^*(N) = \Phi^*(M)$, which implies $N = M$ since Φ is essentially surjective, i.e., S generates M . In particular, if $\Phi^*(M)$ is finitely generated then so is M .

Now suppose that $\Phi^*(M)$ is noetherian. Given a subrepresentation N of M , we obtain a subrepresentation $\Phi^*(N)$ of $\Phi^*(M)$. Since $\Phi^*(M)$ is noetherian, it follows that $\Phi^*(N)$ is finitely generated. Thus N is finitely generated, and so M is noetherian. \square

Corollary 3.4.5. *Let $\Phi: \mathcal{C} \rightarrow \mathcal{C}'$ be an essentially surjective functor satisfying property (F) and suppose $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian. Then $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian.*

Proof. Let M be a finitely generated \mathcal{C}' -module. Then $\Phi^*(M)$ is finitely generated by Proposition 3.4.3, and therefore noetherian, and so M is noetherian by Proposition 3.4.4. \square

3.5. Noetherian posets. Let X be a partially ordered set (poset). Then X satisfies the **ascending chain condition** (ACC) if every ascending chain in X stabilizes, i.e., given $x_1 \leq x_2 \leq \dots$ in X we have $x_i = x_{i+1}$ for $i \gg 0$. The **descending chain condition** (DCC) is defined similarly. An **anti-chain** in X is a sequence x_1, x_2, \dots such that $x_i \not\leq x_j$ for all $i \neq j$. An **ideal** in X is a subset I of X such that $x \in I$ and $x \leq y$ implies $y \in I$. We write $\mathcal{J}(X)$ for the poset of ideals of X , ordered by inclusion. For $x \in X$, the **principal ideal** generated by x is $\{y \mid y \geq x\}$. An ideal is **finitely generated** if it is a finite union of principal ideals. The following result is left to the reader as an exercise.

Proposition 3.5.1. *The following conditions on X are equivalent:*

- (a) *The poset X satisfies DCC and has no infinite anti-chains.*
- (b) *Given a sequence x_1, x_2, \dots in X , there exists $i < j$ such that $x_i \leq x_j$.*
- (c) *The poset $\mathcal{J}(X)$ satisfies ACC.*
- (d) *Every ideal of X is finitely generated.*

The poset X is **noetherian** if the above conditions are satisfied. Where we say “ X is noetherian,” one often sees “ \leq is a well-quasi-order” in the literature. Similarly, where we say “ X satisfies DCC” one sees “ \leq is well-founded.”

Proposition 3.5.2. *Let X be a noetherian poset and let x_1, x_2, \dots be a sequence in X . Then there exists an infinite sequence of indices $i_1 < i_2 < \dots$ such that $x_{i_1} \leq x_{i_2} \leq \dots$.*

Proof. Let I be the set of indices such that $i \in I$ and $j > i$ implies that $x_i \leq x_j$. If I is infinite, then there is $i < i'$ with $i, i' \in I$ such that $x_i \leq x_{i'}$ by definition of noetherian and hence contradicts the definition of I . So I is finite; let i_1 be any number larger than all elements of I . Then by definition of I , we can find $x_{i_1} \leq x_{i_2} \leq \dots$. \square

Proposition 3.5.3. *Let X and Y be noetherian posets. Then $X \times Y$ is noetherian.*

Proof. Let $(x_1, y_1), (x_2, y_2), \dots$ be an infinite sequence in $X \times Y$. Since X is noetherian, there exists $i_1 < i_2 < \dots$ such that $x_{i_1} \leq x_{i_2} \leq \dots$ (Proposition 3.5.2). Since Y is noetherian, there exists $i'_j < i''_j$ such that $y_{i_j} \leq y_{i'_j}$, and hence $(x_{i_j}, y_{i_j}) \leq (x_{i'_j}, y_{i'_j})$. \square

Given a poset X , let X^* be the set of finite words $x_1 \cdots x_n$ with $x_i \in X$. We define $x_1 \cdots x_n \leq x'_1 \cdots x'_m$ if there exist $1 \leq i_1 < \dots < i_n \leq m$ such that $x_j \leq x'_{i_j}$ for $j = 1, \dots, n$.

Theorem 3.5.4 (Higman's lemma [Hi]). *If X is a noetherian poset, then so is X^* .*

Proof. Suppose that X^* is not noetherian. We use Nash-Williams' theory of minimal bad sequences [NW] to get a contradiction. A sequence w_1, w_2, \dots of elements in X^* is bad if $w_i \not\leq w_j$ for all $i < j$. We pick a bad sequence minimal in the following sense: for all $i \geq 1$, among all bad sequences beginning with w_1, \dots, w_{i-1} (this is the empty sequence for $i = 1$), $\ell(w_i)$ is as small as possible. Let $x_i \in X$ be the first element of w_i and let v_i be the subword of w_i obtained by removing x_i . By Proposition 3.5.2, there is an infinite sequence $i_1 < i_2 < \dots$ such that $x_{i_1} \leq x_{i_2} \leq \dots$. Then $w_1, w_2, \dots, w_{i_1-1}, v_{i_1}, v_{i_2}, \dots$ is a bad sequence because $v_{i_j} \leq w_{i_j}$ for all j , and $v_{i_j} \leq v_{i'_j}$ would imply that $w_{i_j} \leq w_{i'_j}$. It is smaller than our minimal bad sequence, so we have reached a contradiction. Hence X^* is noetherian. \square

3.6. Monomial representations and Gröbner bases. Let \mathcal{C} be an essentially small category and let \mathbf{Set} be the category of sets. Fix a functor $S: \mathcal{C} \rightarrow \mathbf{Set}$, and let $P = \mathbf{k}[S]$, i.e., $P(x)$ is the free \mathbf{k} -module on the set $S(x)$.

Given $f \in S(x)$, we write e_f for the corresponding element of $P(x)$. An element of P is a **monomial** if it is of the form λe_f for some $\lambda \in \mathbf{k}$ and $f \in S(x)$. A subrepresentation M of P is **monomial** if $M(x)$ is spanned by the monomials it contains, for all objects x .

To connect arbitrary subrepresentations of P to monomial subrepresentations, we need a theory of monomial orders. Let \mathbf{WO} be the category of well-ordered sets and strictly order-preserving functions. There is a forgetful functor $\mathbf{WO} \rightarrow \mathbf{Set}$. An **ordering** on S is a lifting of S to \mathbf{WO} . More concretely, an ordering on S is a choice of well-order on $S(x)$, for each $x \in \mathcal{C}$, such that for every morphism $x \rightarrow y$ in \mathcal{C} the induced map $S(x) \rightarrow S(y)$ is strictly order-preserving. We write \preceq for an ordering; S is **orderable** if it admits an ordering.

Suppose \preceq is an ordering on S . Given non-zero $\alpha = \sum_{f \in S(x)} \lambda_f e_f$ in $P(x)$, we define the **initial term** of α , denoted $\text{init}(\alpha)$, to be $\lambda_g e_g$, where $g = \max_{\preceq} \{f \mid \lambda_f \neq 0\}$. The **initial variable** of α , denoted $\text{init}_0(\alpha)$, is g . Now let M be a subrepresentation of P . We define the **initial representation** of M , denoted $\text{init}(M)$, as follows: $\text{init}(M)(x)$ is the \mathbf{k} -span of the elements $\text{init}(\alpha)$ for non-zero $\alpha \in M(x)$. The name is justified by the following result.

Proposition 3.6.1. *Notation as above, $\text{init}(M)$ is a monomial subrepresentation of P .*

Proof. Let $\alpha = \sum_{i=1}^n \lambda_i e_{f_i}$ be an element of $M(x)$ with each λ_i non-zero, ordered so that $f_i \prec f_1$ for all $i > 1$. Thus $\text{init}(\alpha) = \lambda_1 e_{f_1}$. Let $g: x \rightarrow y$ be a morphism. Then $g_*(\alpha) = \sum_{i=1}^n \lambda_i e_{g_*(f_i)}$. Since $g_*: S(x) \rightarrow S(y)$ is strictly order-preserving, we have $g_*(f_i) \prec g_*(f_1)$ for all $i > 1$. Thus $\text{init}(g_*(\alpha)) = \lambda_1 e_{g_*(f_1)}$, or, in other words, $\text{init}(g_*(\alpha)) = g_*(\text{init}(\alpha))$. This shows that g_* maps $\text{init}(M)(x)$ into $\text{init}(M)(y)$, and so $\text{init}(M)$ is a subrepresentation of P . That it is monomial follows immediately from its definition. \square

Proposition 3.6.2. *If $N \subseteq M$ are subrepresentations of P and $\text{init}(N) = \text{init}(M)$, then $M = N$.*

Proof. Assume that $M(x) \neq N(x)$ for $x \in \mathcal{C}$. Let $K \subset S(x)$ be the set of all elements which appear as the initial variable of some element of $M(x) \setminus N(x)$. Then $K \neq \emptyset$, so has

a minimal element f with respect to \preceq . Pick $\alpha \in M(x) \setminus N(x)$ with $\text{init}_0(\alpha) = f$. By assumption, there exists $\beta \in N(x)$ with $\text{init}(\alpha) = \text{init}(\beta)$. But then $\alpha - \beta \in M(x) \setminus N(x)$, and $\text{init}_0(\alpha - \beta) \prec \text{init}_0(\alpha)$, a contradiction. Thus $M = N$. \square

Let M be a subrepresentation of P . A set \mathfrak{G} of elements of M is a **Gröbner basis** of M if $\{\text{init}(\alpha) \mid \alpha \in \mathfrak{G}\}$ generates $\text{init}(M)$. Note that M has a finite Gröbner basis if and only if $\text{init}(M)$ is finitely generated. As usual, we have:

Proposition 3.6.3. *Let \mathfrak{G} be a Gröbner basis of M . Then \mathfrak{G} generates M .*

Proof. Let $N \subseteq M$ be the subrepresentation generated by \mathfrak{G} . Then $\text{init}(N)$ contains $\text{init}(\alpha)$ for all $\alpha \in \mathfrak{G}$, and so $\text{init}(N) = \text{init}(M)$. Thus $M = N$ by Proposition 3.6.2. \square

We now come to our main result.

Theorem 3.6.4. *Suppose \mathbf{k} is noetherian, S is orderable, and $|S|$ is noetherian. Then every subrepresentation of P has a finite Gröbner basis. In particular, P is a noetherian object of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$.*

Proof. It suffices to show that every monomial subrepresentation is finitely generated. Let Q be the direct product of the poset of ideals in \mathbf{k} with $|S|$. Suppose we have a strictly increasing chain of monomial subrepresentations $M_1 \subset M_2 \subset \cdots \subset P$. For each i , pick a monomial $\lambda_i e_{f_i} = m_i \in M_i \setminus M_{i-1}$ and let I_i be the ideal in \mathbf{k} generated by all λ_i which are coefficients of e_{f_i} appearing in any monomial in M_i . Then we get a sequence $(I_1, f_1), (I_2, f_2), \dots$ in Q which are incomparable, contradicting the fact that $|S|$, and hence Q (by Proposition 3.5.3), is noetherian. \square

3.7. Gröbner categories. Let \mathcal{C} be an essentially small category. For an object x , let $S_x: \mathcal{C} \rightarrow \mathbf{Set}$ be the functor given by $S_x(y) = \text{Hom}_{\mathcal{C}}(x, y)$. Note that $P_x = \mathbf{k}[S_x]$.

Definition 3.7.1. We say that \mathcal{C} is **Gröbner** if, for all objects x , the functor S_x is orderable and the poset $|S_x|$ is noetherian. We say that \mathcal{C} is **quasi-Gröbner** if there exists a Gröbner category \mathcal{C}' and an essentially surjective functor $\mathcal{C}' \rightarrow \mathcal{C}$ satisfying property (F). \square

Theorem 3.7.2. *Let \mathcal{C} be a quasi-Gröbner category. Then for any noetherian ring \mathbf{k} , the category $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian.*

Proof. First suppose that \mathcal{C} is a Gröbner category. Then every principal projective of $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian, by Theorem 3.6.4, and so $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Proposition 3.4.1.

Now suppose that \mathcal{C} is quasi-Gröbner, and let $\Phi: \mathcal{C}' \rightarrow \mathcal{C}$ be an essentially surjective functor satisfying property (F), with \mathcal{C}' Gröbner. Then $\text{Rep}_{\mathbf{k}}(\mathcal{C}')$ is noetherian, by the previous paragraph, and so $\text{Rep}_{\mathbf{k}}(\mathcal{C})$ is noetherian by Corollary 3.4.5. \square

Remark 3.7.3. If the functor S_x is orderable, then the group $\text{Aut}(x)$ admits a well-order compatible with the group operation, and is therefore trivial. Thus, in a Gröbner category, there are no non-trivial automorphisms. \square

Proposition 3.7.4. *The cartesian product of finitely many (quasi-)Gröbner categories is (quasi-)Gröbner.*

We leave the details to the reader.

3.8. Example: \mathbf{FI}_d -modules. Let d be a positive integer. Define \mathbf{OI}_d to be the ordered version of \mathbf{FI}_d : its objects are totally ordered finite sets and a morphisms between S and T is a pair (f, g) with f an order-preserving injection, and g is an arbitrary function $g: T \setminus f(S) \rightarrow [d]$. When $d = 1$, we will write \mathbf{FI} and \mathbf{OI} instead of \mathbf{FI}_1 and \mathbf{OI}_1 .

Let $\Sigma = \{0, \dots, d\}$, and let \mathcal{L} be the subset of Σ^* consisting of words $w_1 \cdots w_r$ in which exactly n of the w_i are equal to 0. Partially order \mathcal{L} using the subsequence order, i.e., if $s: [i] \rightarrow \Sigma$ and $t: [j] \rightarrow \Sigma$ are words then $s \leq t$ if there exists $I \subseteq [j]$ such that $s = t|_I$.

Lemma 3.8.1. *The poset \mathcal{L} is noetherian.*

Proof. Noetherianity is an immediate consequence of Higman’s lemma (Theorem 3.5.4). \square

Our main result about \mathbf{OI}_d is the following theorem.

Theorem 3.8.2. *The category \mathbf{OI}_d is Gröbner.*

Proof. Let n be a non-negative integer, and regard $x = [n]$ as an object of \mathcal{C} . Pick $(f, g) \in \text{Hom}_{\mathcal{C}}([n], [m])$. Define $h: [m] \rightarrow \Sigma$ to be the function which is 0 on the image of f , and equal to g on the complement of the image of f . One can recover (f, g) from h since f is required to be order-preserving and injective. This construction therefore defines an isomorphism of posets $i: S_x \rightarrow \mathcal{L}$. It follows that $|\mathcal{C}_x|$ is noetherian. Furthermore, putting the lexicographic order on \mathcal{L} (using the standard order on Σ) gives a lift of the functor S_x to a well-ordering. Thus \mathcal{C} is Gröbner. \square

Remark 3.8.3. The results about \mathbf{OI} can be made more transparent with the following observation: the set of order-increasing injections $f: [n] \rightarrow [m]$ is naturally in bijection with monomials in x_0, \dots, x_n of degree $m - n$ by assigning the monomial $\mathbf{m}_f = \prod_{i=0}^n x_i^{f(i+1) - f(i) - 1}$ using the convention $f(0) = 0$ and $f(n + 1) = m + 1$. Given $g: [n] \rightarrow [m']$, there is a morphism $h: [m] \rightarrow [m']$ with $g = hf$ if and only if \mathbf{m}_f divides \mathbf{m}_g . Thus the monomial subrepresentations of P_n are in bijection with monomial ideals in the polynomial ring $\mathbf{k}[x_0, \dots, x_n]$.

In fact, this also shows that finitely generated \mathbf{OI} -modules (and hence finitely generated \mathbf{FI} -modules) have eventually polynomial growth when \mathbf{k} is a field, i.e., the function $d \mapsto \dim_{\mathbf{k}} M([d])$ is a polynomial for $d \gg 0$. This can also be put in the general framework of “lingual categories” as in [SS1], but we will omit that discussion. \square

Theorem 3.8.4. *The forgetful functor $\Phi: \mathbf{OI}_d \rightarrow \mathbf{FI}_d$ satisfies property (F). In particular, \mathbf{FI}_d is quasi-Gröbner.*

Proof. Let $x = [n]$ be a given object of \mathbf{FI}_d . If y is any totally ordered set, then any morphism $f: x \rightarrow y$ can be factored as $x \xrightarrow{\sigma} x \xrightarrow{f'} y$, where σ is a permutation and f' is order-preserving. It follows that we can take $y_1, \dots, y_{n!}$ to all be $[n]$, and $f_i: x \rightarrow \Phi(y_i)$ to be the i th element of the symmetric group S_n (under any enumeration). This establishes the claim. Since \mathbf{OI}_d is Gröbner, this shows that \mathbf{FI}_d is quasi-Gröbner. \square

Corollary 3.8.5. *If \mathbf{k} is left-noetherian then $\text{Rep}_{\mathbf{k}}(\mathbf{FI}_d)$ is noetherian.*

4. HOMOLOGICAL STABILITY FOR SYMMETRIC GROUPS

Some of the applications of representation stability and the theory we’ve been developing was motivated by generalizations of homological stability. So we’ll pause to prove homological

stability for the symmetric groups to get a feeling for what sort of result this is and to illustrate the use of spectral sequences in this subject.

See §A.3 for basic definitions and properties of group homology. We have inclusions of symmetric groups $\Sigma_1 \rightarrow \Sigma_2 \rightarrow \cdots$ which induce maps on homology

$$H_i(\Sigma_1) \rightarrow H_i(\Sigma_2) \rightarrow \cdots$$

for each i , and Nakaoka's stability theorem says that the map $H_i(\Sigma_{n-1}) \rightarrow H_i(\Sigma_n)$ is an isomorphism for $n > 2i$. We will follow Kerz's proof [Ke] of this fact.

4.1. The complex of injective words. Let m be a fixed positive integer. An **injective word** in the alphabet $[m]$ is a sequence (i_1, \dots, i_n) of elements in $[m]$, such that $i_j \neq i_k$ for $j \neq k$. Let $C_n(m)$ be the free abelian group with basis indexed by the injective words of length n in the alphabet $[m]$. By convention, $C_0(m) = \mathbf{Z}$ with 1 basis element given by the empty sequence. We will abuse notation and use (i_1, \dots, i_n) to denote the basis vector in $C_n(m)$. Define a map

$$\begin{aligned} d: C_n(m) &\rightarrow C_{n-1}(m) \\ (i_1, \dots, i_n) &\mapsto \sum_{j=1}^n (-1)^{j+1} (i_1, \dots, \hat{i}_j, \dots, i_n) \end{aligned}$$

where the hat denotes that we are omitting that term. A quick check shows that $d^2 = 0$, so we have a chain complex $C_*(m)$.

A scalar multiple of a sequence is called a **term**. If $x \in C_n(m)$ is a sum of sequences, then a number $i \in [m]$ is said to appear in x if it is an element in one of the sequences. Finally, we define a partially defined product on $C_n(m)$ as follows. If $x \in C_n(m)$ and $x' \in C_\ell(m)$ are terms with $x = N(i_1, \dots, i_n)$ and $x' = M(i'_1, \dots, i'_\ell)$, then we define the product $xx' = NM(i_1, \dots, i_n, i'_1, \dots, i'_\ell)$ assuming that this is still an injective word. We extend this to sums whenever all of the corresponding products make sense. When the product is defined for $x \in C_n(m)$ and $x' \in C_\ell(m)$, it satisfies the Leibniz rule in the sense that

$$d(xx') = d(x)x' + (-1)^n xd(x').$$

Lemma 4.1.1. *If $x \in C_n(m)$ is a cycle and there is a number i not appearing in x , then x is a boundary.*

Proof. By the Leibniz rule, we have $d((i)x) = x - (i)d(x)$. The second quantity is x since $d(x) = 0$. \square

Theorem 4.1.2 (Farmer). $H_i(C_*(m)) = 0$ for $i < m$.

Proof. This is clear for $i = 0$. To prove the rest of the result, we do induction on m , starting with $m = 2$. In that case, we have

$$C_2(2) \rightarrow C_1(2) \rightarrow C_0(2).$$

The kernel of the second map is generated by $(1) - (2) = d(1, 2)$, so $H_1(C_*(2)) = 0$.

Now let $c \in C_n(m)$ be a cycle with $n < m$. If there is some letter not appearing in c , then we can apply Lemma 4.1.1 and conclude c is a boundary. It will suffice to show that there is some letter not appearing in $c - d(w)$ for some $w \in C_{n+1}(m)$, so we now show how to construct such a w .

Suppose there is a letter α that appears as the first entry in some basis element in c . Then write

$$c = \sum_j (\alpha)c_j + c'$$

where the c_j are terms and α does not appear as the first entry in c' . Since $n < m$, for each j there is a letter $\alpha_j \neq \alpha$ that does not appear in c_j . Then we have

$$\begin{aligned} c - d\left(\sum_j (\alpha_j, \alpha)c_j\right) &= c - \sum_j ((\alpha)c_j - (\alpha_j)d((\alpha)c_j)) \\ &= c' + \sum_j ((\alpha_j)c_j - (\alpha_j, \alpha)d(c_j)), \end{aligned}$$

and α does not appear as the first entry in the last expression. By induction on i , we will now show that we can add a boundary to the above expression so that α does not appear in the first i entries.

So suppose α does not appear in the first i entries of the cycle c . Write

$$c = \sum_j s_j(\alpha)c_j + c'$$

where $\ell(s_j) = i$, s_j and c_j do not contain α , the c_j are distinct for different j , and c' does not contain α in the first $i + 1$ entries. Then we have

$$\begin{aligned} 0 = d(c) &= \sum_j (d(s_j)(\alpha)c_j + (-1)^i s_j d((\alpha)c_j)) + d(c') \\ &= \sum_j (d(s_j)(\alpha)c_j + (-1)^i s_j c_j + (-1)^{i+1} s_j(\alpha)d(c_j)) + d(c'). \end{aligned}$$

By our assumption, the only summands where α appears in the i th entry are $d(s_j)(\alpha)c_j$, so $\sum_j d(s_j)(\alpha)c_j = 0$. Furthermore, the c_j are distinct, so we conclude that $d(s_j) = 0$ for all j . Now, $\ell(s_j) = i < m - 1 - \ell(c_j)$ and s_j uses only at most $m - 1 - \ell(c_j)$ letters, so by our induction hypothesis, we conclude there exists s'_j using these same $m - 1 - \ell(c_j)$ letters such that $d(s'_j) = s_j$. In particular, the element

$$z = \sum_j s'_j(\alpha)c_j$$

is well-defined, and we have

$$\begin{aligned} c - d(z) &= c - \sum_j (s_j(\alpha)c_j + (-1)^{i+1} s'_j c_j + (-1)^{i+2} s'_j(\alpha)d(c_j)) \\ &= c' - \sum_j ((-1)^{i+1} s'_j c_j + (-1)^i s'_j(\alpha)d(c_j)). \end{aligned}$$

Again, α does not appear in the first $i + 1$ entries of the last expression. Finally, once we have $i = n$, we have found the desired element $c - d(w)$. \square

4.2. Nakaoka's theorem.

Theorem 4.2.1 (Nakaoka). *The map $H_i(\Sigma_{n-1}) \rightarrow H_i(\Sigma_n)$ is an isomorphism for $n > 2i$.*

Proof. We go by induction on n . For $n = 3$, we only consider $i = 1$, in which case we have

$$H_1(\Sigma_2) = H_1(\Sigma_3) = \mathbf{Z}/2$$

is the abelianization of the group, and the induced map is the identity.

For $n \geq 4$, define a new complex $C'_*(n)$ by $C'_i(n) = C_{i+1}(n)$ for $i \geq 0$. Then we have removed the term $C_0(n)$, so $H_0(C'_*(n)) = \mathbf{Z}$ and $H_i(C'_*(n)) = 0$ for $0 < i < n - 1$.

Note that Σ_n acts on the $C'_i(n)$ by permuting the letters and that the differential d is equivariant for these actions. We can define the homology of a group with coefficients in a chain complex (this is a special case of hyperhomology), but we won't need the precise definition. We just need to know that there are two spectral sequences which both converge to this value [Br, §VII.5]:

$$\begin{aligned} E_{p,q}^2 &= H_p(\Sigma_n; H_q(C'_*(n))) \implies H_{p+q}(\Sigma_n; C'_*(n)) \\ E_{p,q}^1 &= H_q(\Sigma_n; C'_p(n)) \implies H_{p+q}(\Sigma_n; C'_*(n)). \end{aligned}$$

Consider the first spectral sequence. As mentioned before, $H_q(C'_*(n)) = 0$ for $0 < q < n - 1$. Hence the only terms of degree d with $0 \leq d < n - 1$ is $H_d(\Sigma_n; H_0(C'_*(n))) = H_d(\Sigma_n)$. Furthermore, on each page, there is no nonzero differential to these terms, so

$$H_d(\Sigma_n; C'_*(n)) = E_{d,0}^\infty = H_d(\Sigma_n) \quad (0 \leq d < n - 1).$$

Now consider the second spectral sequence. The Σ_n -representation $C'_p(n)$ is the permutation representation on the set of injective words of length $p + 1$. This has a transitive action of Σ_n with stabilizer subgroup Σ_{n-p-1} , so it is isomorphic to the induced module $\text{Ind}_{\Sigma_{n-p-1}}^{\Sigma_n} \mathbf{Z}$, so

$$H_q(\Sigma_n; C'_p(n)) = H_q(\Sigma_{n-p-1})$$

by Shapiro's lemma. The map $C'_p(n) \rightarrow C'_{p-1}(n)$ corresponding to forgetting the j th entry becomes the map

$$H_q(\Sigma_{n-p-1}) \rightarrow H_q(\Sigma_{n-p})$$

induced by the inclusion of Σ_{n-p-1} into Σ_{n-p} as the permutations that fix j . These all differ by conjugation by an element of Σ_{n-p} , so are the same for all j . Hence the differential induces a map on homology which is an alternating sum of the same map, and we see that it is 0 if p is odd and the inclusion map if p is even. Hence the spectral sequence looks like

$$\begin{array}{ccccccc} H_2(\Sigma_{n-1}) & \xleftarrow{0} & H_2(\Sigma_{n-2}) & \xleftarrow{\quad} & H_2(\Sigma_{n-3}) & \xleftarrow{0} & \cdots & H_2(\Sigma_2) & H_2(\Sigma_1) & 0 \\ H_1(\Sigma_{n-1}) & \xleftarrow{0} & H_1(\Sigma_{n-2}) & \xleftarrow{\quad} & H_1(\Sigma_{n-3}) & \xleftarrow{0} & \cdots & H_1(\Sigma_2) & H_1(\Sigma_1) & 0 \\ H_0(\Sigma_{n-1}) & \xleftarrow{0} & H_0(\Sigma_{n-2}) & \xleftarrow{\quad} & H_0(\Sigma_{n-3}) & \xleftarrow{0} & \cdots & H_0(\Sigma_2) & H_0(\Sigma_1) & \mathbf{Z} \end{array}$$

where the maps alternate between being 0 and coming from the standard inclusion.

In the second page, the leftmost column stays the same, i.e., $E_{0,i}^2 = H_i(\Sigma_{n-1})$ for all $i \geq 0$. Now suppose that $n > 2i$. The terms with the same degree i are of the form $H_{i-k}(\Sigma_{n-k-1})$ and so by induction, the maps from the previous symmetric group or to the next symmetric

group are isomorphisms on homology. This implies that the only term of degree i on the E^2 page is $E_{i,0}^2$. Finally, we claim that $E_{0,i}^\infty = E_{0,i}^2$ since the only term that can map to it on the E^r page is $E_{r,i-r+1}^r$, and this is also 0 by the induction hypothesis. In particular, this implies that we have an identification $H_i(\Sigma_{n-1}) = H_i(\Sigma_n)$.

It remains to show that this is induced by the inclusion map, but we will omit this detail since it involves a more detailed analysis of the spectral sequence. \square

Remark 4.2.2. Nakaoka's theorem remains true if we replace \mathbf{Z} by any coefficient ring \mathbf{k} (as long as Σ_n acts by the identity on \mathbf{k}). For example, the proof above is not sensitive to the coefficients we have chosen except in the calculation of $H_1(\Sigma_n)$, but the general statement is that it is $\mathbf{Z}/2 \otimes_{\mathbf{Z}} \mathbf{k}$ for $n \geq 2$. \square

4.3. Twisted homological stability. Let M be an **FI**-module over some coefficient ring \mathbf{k} . Let $M(n)$ denote the value of M on the set $[n]$. This carries an action of the symmetric group Σ_n . Let $\iota: [n] \rightarrow [n+1]$ be the standard inclusion, i.e., $\iota(i) = i$ for all i and let $f: \Sigma_n \rightarrow \Sigma_{n+1}$ also be the standard inclusion. Then for all $\sigma \in \Sigma_n$ and $m \in M(n)$, we have $\iota_*(\sigma m) = f(\sigma)\iota_*(m)$ since $\iota \circ \sigma = f(\sigma)$ as morphisms in **FI**. So by functoriality of group homology, we have induced maps for all i :

$$H_i(\Sigma_n; M(n)) \rightarrow H_i(\Sigma_{n+1}; M(n+1)).$$

When M is the constant functor, i.e., $M_n = \mathbf{k}$ and all morphisms map to the identity, then we considered these maps above and Nakaoka's theorem tells us that they are isomorphisms for $n \gg i$ (more precisely, $n > 2i$). We now extend this to any finitely generated **FI**-module over a noetherian ring \mathbf{k} .

We say that the modules $M(n)$ with the maps $\iota_*: M(n) \rightarrow M(n+1)$ satisfy **twisted homological stability** if the maps on homology are isomorphisms for $n \gg i$.

So now assume that \mathbf{k} is noetherian.

Lemma 4.3.1. *The principal projective module P_d satisfies twisted homological stability.*

Proof. For each n , $P_d(n)$ is the permutation representation on the set of injective words of length d in the alphabet $[n]$. As discussed in the proof of Nakaoka's theorem, this is the induced representation $\text{Ind}_{\Sigma_{n-d}}^{\Sigma_n} \mathbf{k}$ and, via Shapiro's lemma, the map

$$H_i(\Sigma_n; P_d(n)) \rightarrow H_i(\Sigma_{n+1}; P_d(n+1))$$

can be rewritten as

$$H_i(\Sigma_{n-d}; \mathbf{k}) \rightarrow H_i(\Sigma_{n+1-d}; \mathbf{k})$$

which is the map coming from the standard inclusion. Hence twisted homological stability for P_d reduces to Nakaoka's stability theorem. \square

Corollary 4.3.2. *Any finitely generated **FI**-module M satisfies twisted homological stability.*

Proof. Consider the following two statements:

(A_{*i*}): For any finitely generated **FI**-module M , the map

$$H_i(\Sigma_n; M(n)) \rightarrow H_i(\Sigma_{n+1}; M(n+1))$$

is surjective for $n \gg i$.

(B_{*i*}): For any finitely generated **FI**-module M , the map

$$H_i(\Sigma_n; M(n)) \rightarrow H_i(\Sigma_{n+1}; M(n+1))$$

is an isomorphism for $n \gg i$.

We will show that (B_{i-1}) implies (A_i) for all i , and that (A_i) and (B_{i-1}) together imply (B_i) for all i . With the convention that $H_{-1}(\Sigma_n; M(n)) = 0$ for all n , the statements (A_{-1}) and (B_{-1}) are vacuously true.

Let M be a finitely generated **FI**-module generated by homogeneous elements x_1, \dots, x_r and of degrees d_1, \dots, d_r . Let $P = P_{d_1} \oplus \dots \oplus P_{d_r}$ so that we have a canonical surjection $P \rightarrow M$, and let K be the kernel. Then K is finitely generated by the noetherian property for **FI**-modules. Fix i and consider the long exact sequence on homology:

$$\begin{array}{ccccccccc} H_i(\Sigma_n; K(n)) & \longrightarrow & H_i(\Sigma_n; P(n)) & \longrightarrow & H_i(\Sigma_n; M(n)) & \longrightarrow & H_{i-1}(\Sigma_n; K(n)) & \longrightarrow & H_{i-1}(\Sigma_n; P(n)) \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\ H_i(\Sigma_{n+1}; K(n+1)) & \longrightarrow & H_i(\Sigma_{n+1}; P(n+1)) & \longrightarrow & H_i(\Sigma_{n+1}; M(n+1)) & \longrightarrow & H_{i-1}(\Sigma_{n+1}; K(n+1)) & \longrightarrow & H_{i-1}(\Sigma_{n+1}; P(n+1)) \end{array}$$

By Lemma 4.3.1, β and ε are isomorphisms for $n \gg i$.

Suppose (B_{i-1}) is true. Then δ is an isomorphism for $n \gg i$. Then the first part of the four lemma implies that γ is surjective for $n \gg i$. Since M was arbitrary, (A_i) is true.

Now suppose that (A_i) and (B_{i-1}) are both true, then α and γ are surjective, and δ is an isomorphism for $n \gg i$. The second part of the four lemma then says that γ is injective. Since M was arbitrary, (B_i) is true. \square

5. REPRESENTATION STABILITY FOR CONFIGURATION SPACES

5.1. **Definitions.** Let X be a topological space. For an integer $n \geq 0$, we set

$$\text{Conf}_n(X) = \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}.$$

Alternatively, this is the space of injective functions $[n] \rightarrow X$. It will be convenient to define $\text{Conf}_S(X)$ to be the space of injective functions $S \rightarrow X$ for any finite set S .

Given an injection $S \rightarrow T$, we get a continuous function $\text{Conf}_T(X) \rightarrow \text{Conf}_S(X)$ by precomposition. So we have define a functor from the opposite category of **FI** to the category of topological spaces. To get something linear out of it, we just need to choose some (contravariant) functor from the category of topological spaces to the category of modules over some ring. We will consider singular cohomology with coefficients in a commutative ring \mathbf{k} , denoted $H^i(X; \mathbf{k})$ where $i \geq 0$ is some integer. We won't need to know the details of how this is constructed, but here are two key properties:

- (1) $H^*(X; \mathbf{k}) = \bigoplus_{i \geq 0} H^i(X; \mathbf{k})$ has the structure of a graded ring: given homogeneous elements in degrees i and j , their product has degree $i + j$. Furthermore, it is graded-commutative: if $x \in H^i(X; \mathbf{k})$ and $y \in H^j(X; \mathbf{k})$, then $xy = (-1)^{ij}yx$.
- (2) Given a continuous function $f: X \rightarrow Y$, we get a corresponding \mathbf{k} -linear map $f^*: H^i(Y; \mathbf{k}) \rightarrow H^i(X; \mathbf{k})$, and these compose correctly: $(f \circ g)^* = g^* \circ f^*$. Furthermore, taking the sum of these maps over all i , f^* is a ring homomorphism.
- (3) (Künneth formula) If \mathbf{k} is a field, then we have an isomorphism of graded rings

$$H^*(X \times Y; \mathbf{k}) \cong H^*(X; \mathbf{k}) \otimes_{\mathbf{k}} H^*(Y; \mathbf{k}).$$

In particular,

$$H^n(X \times Y; \mathbf{k}) \cong \bigoplus_{i=0}^n H^i(X; \mathbf{k}) \otimes_{\mathbf{k}} H^{n-i}(Y; \mathbf{k}).$$

In particular, every choice of topological space X and nonnegative integer i , we get an **FI**-module with coefficients in \mathbf{k} which sends a set S to $H^i(\text{Conf}_S(X); \mathbf{k})$ and an injection $f: S \rightarrow T$ to the map $f^*: H^i(\text{Conf}_S(X); \mathbf{k}) \rightarrow H^i(\text{Conf}_T(X); \mathbf{k})$ which is induced by the continuous function $\text{Conf}_T(X) \rightarrow \text{Conf}_S(X)$ obtained by precomposing with f . Denote this **FI**-module by $H^i(\text{Conf}_*(X); \mathbf{k})$.

From what we've discussed so far, a natural question to ask is: for which X and i is this **FI**-module finitely generated? We won't discuss the most general results known, but instead content ourselves with an easy to analyze case.

Example 5.1.1. Let's consider the case $X = \mathbf{R}^d$. The cohomology ring of $\text{Conf}_n(\mathbf{R}^d)$ is the graded-commutative \mathbf{k} -algebra generated by variables $w_{a,b}$ of degree $d-1$ with $1 \leq a \neq b \leq n$ and modulo the relations:

- $w_{a,b} = (-1)^d w_{b,a}$,
- $w_{a,b}^2 = 0$,
- $w_{a,b}w_{a,c} + w_{b,c}w_{b,a} + w_{c,a}w_{c,b} = 0$ for a, b, c distinct.

The $w_{a,b}$ are elements of $H^{d-1}(\text{Conf}_n(\mathbf{R}^d); \mathbf{k})$.

Given an injection $f: \{1, \dots, n\} \rightarrow \{1, \dots, m\}$, the action of f^* is to replace each $w_{a,b}$ with $w_{f(a),f(b)}$. Note that this is well-defined: f^* sends each of the relations to another relation for the cohomology ring of $\text{Conf}_m(\mathbf{R}^d)$.

Assume now that $d > 1$. We can see directly that $H^{d-1}(\text{Conf}_*(\mathbf{R}^d); \mathbf{k})$ is finitely generated: it is either the permutation representation on the set of 2-element subsets of $[n]$ or a twist of this by the sign character, so is generated by $w_{1,2} \in H^{d-1}(\text{Conf}_2(\mathbf{R}^d); \mathbf{k})$. In general, $H^{i(d-1)}(\text{Conf}_*(\mathbf{R}^d); \mathbf{k})$ is a quotient of $(H^{d-1}(\text{Conf}_*(\mathbf{R}^d); \mathbf{k}))^{\boxtimes i}$ and hence is finitely generated (see Exercise 5.1.2).

If $m = 1$ and \mathbf{k} is a field, one can show that $H^0(\text{Conf}_n(\mathbf{R}); \mathbf{k})$ has dimension $n!$, and hence cannot be a finitely generated **FI**-module. This can either be done using the relations above, or noting that $\text{Conf}_n(\mathbf{R})$ has $n!$ connected components, each one indexed by the order in which the points (x_1, \dots, x_n) appear on the line, and the dimension of H^0 is the number of connected components. □

Exercise 5.1.2. If M and N are **FI**-modules over \mathbf{k} , then define an **FI**-module $M \boxtimes N$, the **pointwise (or Segre) tensor product** as follows. On objects, we set $(M \boxtimes N)(S) = M(S) \otimes_{\mathbf{k}} N(S)$, and given an injection $f: S \rightarrow T$, we set $M(S) \otimes_{\mathbf{k}} N(S) \rightarrow M(T) \otimes_{\mathbf{k}} N(T)$ to be the tensor product of the maps $M(f) \otimes N(f)$.

If M and N are finitely generated, show that $M \boxtimes N$ is also finitely generated.

Hint: Reduce to the case when M and N are principal projectives. □

5.2. A spectral sequence. For this part, see [To, §2]. Let X be a real orientable manifold of dimension d and let \mathbf{k} be a field. Fix a positive integer n . Given $1 \leq a \leq n$, let $p_a: X^n \rightarrow X$ be the projection onto the a th factor, i.e., $p_a(x_1, \dots, x_n) = x_a$. Similarly, given $a \neq b$, let $p_{a,b}: X^n \rightarrow X^2$ be the map $p_{a,b}(x_1, \dots, x_n) = (x_a, x_b)$.

Define a bigraded algebra $A(n)$ over $H^*(X^n; \mathbf{k})$ with variables $G_{a,b}$ ($1 \leq a \neq b \leq n$) modulo the relations

- $G_{a,b} = (-1)^d G_{b,a}$,
- $G_{a,b}^2 = 0$,
- $G_{a,b}G_{a,c} + G_{b,c}G_{b,a} + G_{c,a}G_{c,b} = 0$ for a, b, c distinct,
- $p_a^*(x)G_{a,b} = p_b^*(x)G_{a,b}$ for $a \neq b$ and $x \in H^*(X; \mathbf{k})$.

Here the bidegree of $H^i(X^n; \mathbf{k})$ is $(i, 0)$ and the bidegree of $G_{a,b}$ is $(0, d-1)$.

Let $A(n)_{i,j}$ be the space of bidegree (i, j) elements in $A(n)$. As in [To, Theorem 1], there is a cohomology spectral sequence

$$E_d^{p,q} = A(n)_{p,q} \implies H^{p+q}(\text{Conf}_n(X); \mathbf{k})$$

We won't discuss the differentials in this spectral sequence. All we will need to know is that the terms $A(n)_{p,q}$, as n varies, form an **FI**-module, and that this spectral sequence can be upgraded to a spectral sequence of **FI**-modules.

First, given an injection $f: [n] \rightarrow [m]$, we get a continuous function $X^m \rightarrow X^n$ given by $(x_1, \dots, x_m) \mapsto (x_{f(1)}, \dots, x_{f(n)})$, and hence a corresponding map on cohomology $f^*: H^*(X^n; \mathbf{k}) \rightarrow H^*(X^m; \mathbf{k})$. Using the Künneth formula $H^*(X^n; \mathbf{k}) \cong H^*(X; \mathbf{k})^{\otimes n}$, this can be described as follows. Let $e \in H^0(X; \mathbf{k})$ be the unit for multiplication. Then $\sum v_1 \otimes \dots \otimes v_n$ gets sent to $\sum w_1 \otimes \dots \otimes w_m$ where $w_i = e$ if i is not in the image of f , and $w_{f(j)} = v_j$ otherwise.

Say that X is of **finite type** if $H^*(X; \mathbf{k})$ is a finite-dimensional vector space. This holds if X is compact, or Euclidean space, for instance.

Lemma 5.2.1. *If X is a finite type and connected manifold, then for each i , the **FI**-module $[n] \mapsto H^i(X^n; \mathbf{k})$ is finitely generated.*

Proof. Since X is connected, $H^0(X; \mathbf{k})$ is 1-dimensional with basis e . By the Künneth formula,

$$H^i(X^n; \mathbf{k}) = \bigoplus_{\substack{(j_1, \dots, j_n) \in \mathbf{Z}_{\geq 0}^n \\ i = j_1 + \dots + j_n}} H^{j_1}(X; \mathbf{k}) \otimes \dots \otimes H^{j_n}(X; \mathbf{k}).$$

At most i of the j_k can be positive, and we claim that the **FI**-module in question is generated in degrees $\leq i$. The point is that any element is a sum of simple tensors, and if there $n > i$, then at least one of the factors must be a multiple of e . So it comes from applying an injection from strictly smaller degree. By the finite type assumption, $H^i(X^n; \mathbf{k})$ is finite-dimensional for all n , so $\sum_{n \leq i} \dim H^i(X^n; \mathbf{k}) < \infty$. \square

Theorem 5.2.2 (Church). *If X is a finite type, orientable, connected manifold of dimension $d \geq 2$, then for each i , the **FI**-module $H^i(\text{Conf}_*(X); \mathbf{k})$ is finitely generated.*

Proof. Combining the previous lemma with Example 5.1.1, we see that $[n] \mapsto A(n)_{p,q}$ is a finitely generated **FI**-module for all p, q . The spectral sequence above is compatible with the **FI**-structure, and hence the **FI**-module $H^i(\text{Conf}_*(X); \mathbf{k})$ has a finite filtration whose quotients are computed by the spectral sequence. Namely, they are obtained from the $A(*)_{p,q}$ by taking homology. By the noetherian property for **FI**-modules, each time we take homology (which is the quotient of a submodule), we get another finitely generated **FI**-module, and so $H^i(\text{Conf}_*(X); \mathbf{k})$ itself is a finitely generated **FI**-module. \square

Next, the symmetric group Σ_n acts on $\text{Conf}_n(X)$ by permuting the points, and this action is free (no point has a nontrivial stabilizer). Let

$$\text{UConf}_n(X) = \text{Conf}_n(X) / \Sigma_n$$

be the quotient space. Its points are unordered n -tuples of points in X . We have a Cartan–Leray spectral sequence (dual of [Mc, §8^{bis}.2]):

$$(5.2.3) \quad E_2^{p,q} = H^p(\Sigma_n; H^q(\text{Conf}_n(X); \mathbf{k})) \implies H^{p+q}(\text{UConf}_n(X); \mathbf{k}).$$

If $n!$ is invertible in \mathbf{k} , then $H^p(\Sigma_n; H^q(\text{Conf}_n(X); \mathbf{k})) = 0$ whenever $p > 0$, and so there are no nontrivial differentials in the spectral sequence, so we get the following result.

Proposition 5.2.4. *If $n!$ is invertible in \mathbf{k} , then $H^i(\text{Conf}_n(X); \mathbf{k})^{\Sigma_n} = H^i(\text{UConf}_n(X); \mathbf{k})$.*

Proposition 5.2.5. *Given any **FI**-module M , the direct sum $\bigoplus_{n \geq 0} (M_n)_{\Sigma_n}$ has the structure of a $\mathbf{k}[t]$ -module where multiplication by t on $(M_n)_{\Sigma_n}$ is the action of any injection $[n] \rightarrow [n+1]$. If M is finitely generated, then so is this $\mathbf{k}[t]$ -module.*

Corollary 5.2.6 (Church). *If X is a finite type, orientable, connected manifold of dimension ≥ 2 , then for each i , the function $n \mapsto \dim_{\mathbf{Q}} H^i(\text{UConf}_n(X); \mathbf{Q})$ is constant for $n \gg 0$.*

Proof. We have $H^i(\text{Conf}_n(X); \mathbf{Q})^{\Sigma_n} = H^i(\text{UConf}_n(X); \mathbf{Q})$ by Proposition 5.2.4. By semisimplicity, we also have

$$\dim_{\mathbf{Q}} H^i(\text{Conf}_n(X); \mathbf{Q})^{\Sigma_n} = \dim_{\mathbf{Q}} H^i(\text{Conf}_n(X); \mathbf{Q})_{\Sigma_n}.$$

Now the result follows from Proposition 5.2.5 and the structure theorem for finitely generated modules over $\mathbf{k}[t]$. \square

Naturally, one can ask what happens for fields of positive characteristic. Unfortunately, we have only proven a result about the Σ_n -homology of an **FI**-module, and not its cohomology, which behaves very differently. The situation can be analyzed, though it requires a finer understanding of the structure of **FI**-modules. This might be discussed later in the semester, time permitting. See [NS] for details.

6. ALGEBRAIC GEOMETRY FROM TENSORS

6.1. Review of Zariski topology. Let R be a commutative ring. Let $\text{Spec}(R)$ be the set of prime ideals in R . Given an ideal $I \subset R$, let $V(I) = \{\mathfrak{p} \in \text{Spec}(R) \mid \mathfrak{p} \supseteq I\}$. $\text{Spec}(R)$ is a topological space if we declare the subsets of the form $V(I)$ to be closed, as follows from this list of properties:

- $\bigcap_{s \in S} V(I_s) = V(\sum_{s \in S} I_s)$,
- $V(IJ) = V(I) \cup V(J)$,
- $V(0) = \text{Spec}(R)$,
- $V(R) = \emptyset$.

This is the **Zariski topology** on $\text{Spec}(R)$.

Note that $V(I) = V(\sqrt{I})$ where $\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some } n > 0\}$ is the radical of I . Also, the only primes such that $\{\mathfrak{p}\}$ are closed sets are maximal ideals. We define a variant: $\text{MSpec}(R)$ is the set of maximal ideals with a similarly defined topology. This is usually not well-behaved, though it will be useful in some things we do.

We will be primarily concerned with the case that $R = \mathbf{k}[x_1, \dots, x_n]$ where \mathbf{k} is an algebraically closed field. In this case, Hilbert's nullstellensatz tells us that the maximal ideals of R are all of the form $(x_1 - \alpha_1, \dots, x_n - \alpha_n)$ where $\alpha_i \in \mathbf{k}$. So geometrically, we can think of $\text{Spec}(R)$ as the vector space \mathbf{k}^n with some additional points coming from non-maximal prime ideals. The picture of \mathbf{k}^n usually gives good intuition, so if you haven't seen algebraic geometry before, it will usually be enough to rely on it. We call sets of the form $V(I)$ **algebraic varieties**, or just varieties.

Our main concern will be with recovering the ideal I that defines a closed subset $V(I)$. However, by the above remark, I cannot be recovered since $V(I) = V(\sqrt{I})$. Hence we will be concerned with finding \sqrt{I} , which is well-defined.

If f_1, \dots, f_r are polynomials generating I ,⁴ then in the \mathbf{k}^n model, the closed points of $V(I)$ are those points $(\alpha_1, \dots, \alpha_n)$ such that $f_i(\alpha_1, \dots, \alpha_n) = 0$ for all i . Hence we will say that f_1, \dots, f_r define $V(I)$ set-theoretically (since they determine it as a subset), and we will say they generate the full ideal of $V(I)$ if $I = \sqrt{I}$ is radical. The meaning of this is that any function that vanishes on all closed points of $V(I)$ must be in I .

Remark 6.1.1. In the case $\mathbf{k} = \mathbf{C}$ is the field of complex numbers, we can also put the Euclidean topology on \mathbf{C}^n . Given polynomials f_1, \dots, f_r , we get a continuous function $\mathbf{C}^n \rightarrow \mathbf{C}^r$ defined by $(\alpha_1, \dots, \alpha_n) \mapsto (f_1(\alpha), \dots, f_r(\alpha))$, and so the preimage of 0 is closed in the Euclidean topology. This implies that the Euclidean topology is a refinement of the Zariski topology. In particular, the Zariski closure of a set always contains the Euclidean closure, and in particular, is closed under limits. One can make the notion of being closed under limits precise outside of the case $\mathbf{k} = \mathbf{C}$, though we won't make much use of it. \square

6.2. Border rank. Let V_1, \dots, V_n be vector spaces over an algebraically closed field \mathbf{k} and $\mathbf{V} = V_1 \otimes \dots \otimes V_n$. An element of \mathbf{V} of the form $v_1 \otimes \dots \otimes v_n$ with $v_i \in V_i$ is a simple tensor, and a general element has rank $\leq r$ if it can be expressed as a sum of r simple tensors. Finally, an element in the Zariski closure of rank $\leq r$ tensors has border rank $\leq r$. They form an algebraic variety.

Example 6.2.1. The following example illustrates why the notion of border rank is needed, i.e., why tensor rank is not semicontinuous.

Let $n = 3$ and $\dim V_i = 2$, and for simplicity, take $\mathbf{k} = \mathbf{C}$. Pick bases $\{a_1, a_2\}, \{b_1, b_2\}, \{c_1, c_2\}$ of V_1, V_2, V_3 , respectively. The element

$$\begin{aligned} v &= a_1 \otimes b_1 \otimes c_1 + a_1 \otimes b_1 \otimes c_2 + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \\ &= a_1 \otimes b_1 \otimes (c_1 + c_2) + a_1 \otimes b_2 \otimes c_1 + a_2 \otimes b_1 \otimes c_1 \end{aligned}$$

has rank ≤ 3 , and it can be shown that it does not have rank ≤ 2 , so the rank is exactly 3. However,

$$v = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} ((\varepsilon - 1)a_1 \otimes b_1 \otimes c_1 + (a_1 + \varepsilon a_2) \otimes (b_1 + \varepsilon b_2) \otimes (c_1 + \varepsilon c_2))$$

shows that v has border rank ≤ 2 . \square

Our first main goal is to prove the following theorem:

Theorem 6.2.2 (Draisma–Kuttler). *For each r , there is a constant $C(r)$ such that the variety of border rank $\leq r$ tensors is cut out by polynomials of degree $\leq C(r)$. The constant is independent of n and also the V_i .*

Example 6.2.3. A familiar case is when $n = 2$. Then \mathbf{V} can be thought of as the space of $a \times b$ matrices where $a = \dim V_1$ and $b = \dim V_2$. In this case, a simple tensor is a rank 1 matrix, and rank r in the tensor sense corresponds to rank r in the matrix sense. Border rank $\leq r$ is equivalent to rank $\leq r$. The polynomials in this case are the determinants of all $(r+1) \times (r+1)$ submatrices, and so they are of degree $r+1$. When $n \geq 3$, the situation is more complicated. \square

Remark 6.2.4. For concreteness, consider the case $r = 1$ and $n = 3$ (the case of general n follows in the same way, though the indexing is more cumbersome). Let V_1, V_2, V_3 be vector

⁴ $\mathbf{k}[x_1, \dots, x_n]$ is noetherian by the Hilbert basis theorem, so we can always take r finite

spaces of dimensions d_1, d_2, d_3 and pick a basis e_1, \dots, e_{d_i} for V_i . Let $e_{i,j,k} = e_i \otimes e_j \otimes e_k$, which forms a basis as we vary i, j, k , and let $x_{i,j,k}$ be the dual basis. Suppose we're given a rank 1 tensor of the form

$$v = \left(\sum_i \lambda_i e_i \right) \otimes \left(\sum_j \mu_j e_j \right) \otimes \left(\sum_k \nu_k e_k \right).$$

Then $x_{i,j,k}(v) = \lambda_i \mu_j \nu_k$. In particular, we have the relations

$$x_{i,j,k} x_{i',j',k'} = x_{a,b,c} x_{a',b',c'}$$

whenever $\{i, i'\} = \{a, a'\}$, $\{j, j'\} = \{b, b'\}$, and $\{k, k'\} = \{c, c'\}$. We single out a particular class of these:

$$(6.2.4a) \quad x_{i,j,k} x_{i',j',k'} = x_{\min(i,i'), \min(j,j'), \min(k,k')} x_{\max(i,i'), \max(j,j'), \max(k,k')}.$$

By replacing any product of two variables with the product on the right side, we conclude that every polynomial of degree m on the rank 1 locus is the span of monomials of the form

$$x_{i_1, j_1, k_1} \cdots x_{i_m, j_m, k_m} \quad (i_1 \leq \cdots \leq i_m, \quad j_1 \leq \cdots \leq j_m, \quad k_1 \leq \cdots \leq k_m).$$

The conditions on the indices are independent of one another, so we see that this is the same as the dimension of $\text{Sym}^m(V_1^*) \otimes \text{Sym}^m(V_2^*) \otimes \text{Sym}^m(V_3^*)$ (for a bijection, send the product above to $(e_{i_1}^* \cdots e_{i_m}^*) \otimes (e_{j_1}^* \cdots e_{j_m}^*) \otimes (e_{k_1}^* \cdots e_{k_m}^*)$). One can show that these monomials are linearly independent as functions on the rank 1 locus, so we see that the equations (6.2.4a) generate the full ideal vanishing on the rank 1 locus.

Unfortunately, this reasoning does not readily extend to the $r = 2$ case, though the equations are known in that case via other methods, see Remark 6.4.3. \square

Remark 6.2.5. Part of the reason that the case $n > 2$ is so hard to study is because the notions of border rank and rank behave so differently, and it is not even clear what the precise relation of the two are (does a function of one bound the other for all tensors?). \square

The general idea is to show that there is a way to take the limit as the parameters n and $\dim V_i$ go to ∞ and then to study the resulting limit space. Border rank is preserved upon change of basis (with respect to each V_i) so there is a symmetry that can be used in this problem. The existence of the constant can be then deduced from an equivariant noetherianity property of the limit space.

6.3. Hillar–Sullivant theorem. We'll make use of a theorem of Hillar and Sullivant from [HS]. We're going to use it as a tool for proving Theorem 6.2.2, but see their paper for some other applications.

Let $\text{Inc}(\mathbf{Z}_{>0})$ be the set of strictly increasing functions $\mathbf{Z}_{>0} \rightarrow \mathbf{Z}_{>0}$ ($\mathbf{Z}_{>0}$ is the set of positive integers). It is a monoid under composition. Fix a positive integer d and a commutative ring \mathbf{k} . Define

$$R = \mathbf{k}[x_{i,j} \mid 1 \leq i \leq d, j \in \mathbf{Z}_{>0}].$$

Remark 6.3.1. When $\mathbf{k} = \mathbf{C}$, we have studied this ring in the context of \mathbf{FI}_d -modules. Namely, \mathbf{FI}_d -modules are equivalent, via Schur–Weyl duality, to R -modules with a compatible action of $\mathbf{GL}_\infty(\mathbf{C})$. \square

We study the action of $\text{Inc}(\mathbf{Z}_{>0})$ on R by $f(x_{i,j}) = x_{i,f(j)}$.

Theorem 6.3.2. *Suppose \mathbf{k} is a noetherian ring. Given an increasing chain of $\text{Inc}(\mathbf{Z}_{>0})$ -invariant ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq R$, we have $I_j = I_{j+1}$ for $j \gg 0$.*

Equivalently, any ideal of R that is closed under the action of $\text{Inc}(\mathbf{Z}_{>0})$ is finitely generated under multiplication by R and the action of $\text{Inc}(\mathbf{Z}_{>0})$.

Proof. This can be proven using the techniques from §3, so we'll just outline the steps and leave the details to the reader.

- (1) We define a term order on the variables by first setting

$$x_{1,1} < x_{2,1} < \cdots < x_{d,1} < x_{1,2} < x_{2,2} < \cdots < x_{d,2} < x_{1,3} < \cdots .$$

Now order the monomials using lexicographic order. Then given two monomials $m < m'$, we have $nm < nm'$ for any monomial n , and $fm < fm'$ for any $f \in \text{Inc}(\mathbf{Z}_{>0})$. In particular, we may define an initial ideal for any ideal I closed under $\text{Inc}(\mathbf{Z}_{>0})$ and it suffices to prove that it is finitely generated under the simultaneous action of R and $\text{Inc}(\mathbf{Z}_{>0})$.

- (2) Define a poset P from the monomials in R by $m \preceq m'$ if the ideal generated by m contains m' , i.e., there exists a monomial n and $f \in \text{Inc}(\mathbf{Z}_{>0})$ such that $m' = nf(m)$. We can give a model for P as follows. Given a monomial m , let p be the largest index such that $x_{i,p}$ has nonzero exponent for some i , and define $w(m) = (w_1, \dots, w_p)$ where $w_i \in \mathbf{Z}_{\geq 0}^d$ records the exponents of $x_{1,i}, \dots, x_{d,i}$. If we partially order $\mathbf{Z}_{\geq 0}^d$ by $(\alpha_1, \dots, \alpha_d) \leq (\beta_1, \dots, \beta_d)$ if $\alpha_i \leq \beta_i$ for all i , then the order on $(\mathbf{Z}_{\geq 0}^d)^*$ that we get is exactly the Higman order, so it is a noetherian poset by Theorem 3.5.4.
- (3) Combining the above two steps, we get the theorem for the case when \mathbf{k} is a field. For the general case, we can use the argument in the proof of Theorem 3.6.4. \square

6.4. Proof of Theorem 6.2.2. The outline of the proof is as follows:

- (1) Reduce to the case that $\dim V_i = r + 1$ for all i .
- (2) Construct an infinite limit space $V^{\otimes \infty}$ together with an action of a group G_∞ . Also construct a limit of the border rank $\leq r$ elements $X^{\leq r}$.
- (3) Show that G -equivariant closed subsets of an auxiliary space $Y^{\leq r}$, which contains $X^{\leq r}$, satisfy the descending chain condition.
- (4) Translate the above properties into the desired theorem.

6.4.1. Reduction to bounded number of variables. First, we need the notion of a flattening: given a subset $S \subseteq [n]$, set $U = \bigotimes_{i \in S} V_i$ and $U' = \bigotimes_{i \notin S} V_i$. Then $\mathbf{V} = U \otimes U'$, but now we can identify it with the space of matrices. Given $\omega \in \mathbf{V}$, the corresponding matrix is called a **flattening** of ω (it depends on S). If ω has rank r , then its flattening is a sum of r rank 1 matrices and hence has rank $\leq r$.

Lemma 6.4.1. *It suffices to prove Theorem 6.2.2 when $\dim V_i = r + 1$.*

Proof. We claim that for any $\omega \in \mathbf{V}$, its rank is at most r if and only if for each set of linear maps $\varphi_i: V_i \rightarrow \mathbf{k}^{r+1}$, the rank of its image is also at most r . The “if” direction is clear: if ω has rank $\leq r$, then so are all of its images.

To prove the other direction, suppose that ω has rank $\geq r + 1$. We show that for each i , there exists a linear map $\varphi: V_i \rightarrow \mathbf{k}^{r+1}$ such that the image of ω has rank $\geq r + 1$. Without loss of generality, we may assume $i = 1$. We can reinterpret ω as a linear map

$$V_2^* \otimes \cdots \otimes V_n^* \rightarrow V_1,$$

and we let W be the image. If $\dim W \leq r + 1$, let \mathbf{k}^{r+1} be any subspace of V_1 containing W and let $\varphi: V_1 \rightarrow \mathbf{k}^{r+1}$ be a projection. Then the image of ω has the same rank as ω . Otherwise, if $\dim W > r + 1$, let $\varphi: V \rightarrow \mathbf{k}^{r+1}$ be any linear map such that φ maps W surjectively onto \mathbf{k}^{r+1} . The flattening of the image of ω is then $V_2^* \otimes \cdots \otimes V_n^* \rightarrow \mathbf{k}^{r+1}$ and has rank $r + 1$, so the rank of the image of ω is $\geq r + 1$.

Given the claim, if there is a constant that works for Theorem 6.2.2, then it will work in general: for any tensor product: we can pullback the equations that cut out the rank $\leq r$ locus in $(\mathbf{k}^{r+1})^{\otimes n}$ along all possible choices of linear maps to get equations for the rank $\leq r$ locus in $V_1 \otimes \cdots \otimes V_n$. \square

Exercise 6.4.2. Consider the rank 1 case of flattenings. Show that the equations (6.2.4a) are in the linear span of the 2×2 minors of the flattenings. Deduce that the flattening equations generate the full ideal of the locus of rank 1 tensors. \square

Remark 6.4.3. In fact for the rank 2 case, the 3×3 minors of the flattenings generate the full ideal of the locus of border rank ≤ 2 tensors, at least when the field is of characteristic 0. This is proven in [Ra]. \square

6.4.2. *Spaces of infinite tensors.* In particular, we may as well fix a single space V of dimension $r + 1$ to replace all V_i . Let $X_p^{\leq r} \subset V^{\otimes p}$ be the variety of border rank $\leq r$ tensors, and let $Y_p^{\leq r} \subset V^{\otimes p}$ be the variety of tensors with flattening rank $\leq r$ under all possible flattenings. Then $X_p^{\leq r} \subseteq Y_p^{\leq r}$.

We also fix a linear functional $x_0 \in V^*$ and use this to define maps

$$V^{\otimes(p+1)} \rightarrow V^{\otimes p}$$

$$v_1 \otimes \cdots \otimes v_p \otimes v_{p+1} \mapsto x_0(v_{p+1})v_1 \otimes \cdots \otimes v_p.$$

If $\omega \in V^{\otimes(p+1)}$ has rank $\leq r$, then so does its image in $V^{\otimes p}$. A similar statement holds for flattening rank. Now set

$$V^{\otimes \infty} := \varprojlim_p V^{\otimes p}$$

$$X^{\leq r} := \varprojlim_p X_p^{\leq r}$$

$$Y^{\leq r} := \varprojlim_p Y_p^{\leq r}.$$

Call $Y^{\leq r}$ the **flattening variety**. If we let T_p denote the ring of polynomial functions on $V^{\otimes p}$, then the maps $V^{\otimes(p+1)} \rightarrow V^{\otimes p}$ induce maps $T_p \rightarrow T_{p+1}$ via pullback of functions, and we set

$$T_\infty = \bigcup_p T_p.$$

We will think of this as the ring of polynomial functions on $V^{\otimes \infty}$, or its coordinate ring. We can describe this more concretely: T_p is the symmetric algebra on $(V^*)^{\otimes p}$, so pick a basis x_0, \dots, x_r for V^* (same x_0 as above). Then this naturally gives a basis $x_I = x_{i_1} \otimes \cdots \otimes x_{i_p}$ for $(V^*)^{\otimes p}$ ranging over all $I = (i_1, \dots, i_p) \in \{0, \dots, r\}^p$. We identify this with an infinite word in $\{0, \dots, r\}$ by appending infinitely many 0's at the end. The support of this infinite word is the set of indices with nonzero value. Then T_∞ is the polynomial ring in the x_I as I ranges over all infinite words with finite support.

The space $V^{\otimes p}$ carries an action of Σ_p by permuting tensor factors:

$$\sigma \cdot \sum v_1 \otimes \cdots \otimes v_p = \sum v_{\sigma^{-1}(1)} \otimes \cdots \otimes v_{\sigma^{-1}(p)}.$$

It also has an action of $\mathbf{GL}(V)^p$ by linear change of coordinates:

$$(g_1, \dots, g_p) \cdot \sum v_1 \otimes \cdots \otimes v_p = \sum (g_1 v_1) \otimes \cdots \otimes (g_p v_p).$$

So the semidirect product

$$G_p := \Sigma_p \ltimes \mathbf{GL}(V)^p$$

acts on $V^{\otimes p}$. We have inclusions $G_p \subset G_{p+1}$ compatible with the projection with respect to x_0 , so the union

$$G_\infty = \bigcup_p G_p$$

acts on $V^{\otimes \infty}$. Furthermore, $X^{\leq r}$ is closed under the action of G_∞ .

Both $Y^{\leq r}$ and $X^{\leq r}$ are closed under the action of G_∞ . Furthermore, $Y^{\leq r}$ is (set-theoretically) defined by taking the determinants of all $(r+1) \times (r+1)$ submatrices of all flattenings, which are in particular polynomials of degree $r+1$.

We will need a more precise version of this statement. Let $\mathbf{w} = (w_1, \dots, w_\ell)$ be an ℓ -tuple of distinct infinite words in $\{0, \dots, p\}$ with finite support, and let $\mathbf{w}' = (w'_1, \dots, w'_m)$ be another one. Assume that the support of w_i is disjoint from the support of w'_j for all i, j . Let $M[\mathbf{w}; \mathbf{w}']$ be the $\ell \times m$ matrix whose (i, j) -entry is $x_{w_i + w'_j}$. This is a submatrix of some flattening along subsets I, J where I contains the supports of all w_i and J contains the supports of all w'_j .

Remark 6.4.4. In Remark 6.2.4, we saw that the ideal of the locus of rank 1 tensors is generated by polynomials of degree 2. Suppose we're dealing with the tensor product $\mathbf{V} = V_1 \otimes \cdots \otimes V_n$. By iterating Cauchy's identity (Corollary 2.12.3), we see that, in characteristic 0, the space of quadratic polynomials on \mathbf{V} has a $\mathbf{GL}(V_1) \times \cdots \times \mathbf{GL}(V_n)$ -equivariant decomposition

$$\mathrm{Sym}^2(V_1^* \otimes \cdots \otimes V_n^*) = \bigoplus F_1(V_1^*) \otimes \cdots \otimes F_n(V_n^*)$$

where each F_i is either Sym^2 or \bigwedge^2 , and the sum is over all possible ways for an even number of the F_i to be \bigwedge^2 . The equations themselves span the subspace where a positive (and even) number of the F_i are \bigwedge^2 .

Hence, as n increases, the number of irreducible representations goes to infinity. Note that the statement above is not saying anything about all equations forming finitely many G_∞ -orbits, just that one can do this up to taking radical. In fact, one can show that it is not possible to realize all of the equations using just finitely many G_∞ -orbits, see [Dr1, §7] and also for a discussion of a possible fix. \square

Theorem 6.4.5. *For each r , there is a finite list of pairs $(\mathbf{w}, \mathbf{w}')$ of pairs of words as above of length $r+1$ such that the flattening variety $Y^{\leq r}$ is set-theoretically defined by the G_∞ -orbits of $\det M[w; w']$.*

We omit the proof since it is involved and uses some algebraic geometry which we will not discuss. See [DrK, §4] for the details.

6.4.3. *Equivariant noetherianity of $Y^{\leq r}$.* A topological space X is **noetherian** if for every descending chain of closed subsets $Y_1 \supseteq Y_2 \supseteq \cdots$, we have $Y_i = Y_{i+1}$ for $i \gg 0$.

Example 6.4.6. Noetherianity is not a property of “typical” topological spaces, for example, take $Y_i = \{n \in \mathbf{Z} \mid n \geq i\}$, considered as subsets of \mathbf{R} with the Euclidean topology. \square

Noetherianity is ubiquitous in algebraic geometry. For example, \mathbf{k}^n is noetherian under the Zariski topology since a descending chain of closed subsets is of the form

$$V(I_1) \supseteq V(I_2) \supseteq \cdots$$

for ideals $I_1 \subseteq I_2 \subseteq \cdots \subseteq \mathbf{k}[x_1, \dots, x_n]$. Hilbert’s basis theorem implies that every ideal is finitely generated. In particular, $I = \bigcup_j I_j$ is finitely generated, and these generators must live in some I_j , which implies that $I_j = I_{j+k}$ for any $k \geq 0$. Note, however, that noetherianity of the space $\mathbf{k}[x_1, \dots, x_n]$ is strictly weaker than the statement of Hilbert’s basis theorem, since (when \mathbf{k} is algebraically closed) it only implies that $\sqrt{I_j} = \sqrt{I_{j+1}}$ for $j \gg 0$.

If we have a monoid G acting continuously on X , then we say that X is **G -noetherian** if for every descending chain of G -invariant closed subsets $Y_1 \supseteq Y_2 \supseteq \cdots$, we have $Y_i = Y_{i+1}$ for $i \gg 0$.

Exercise 6.4.7. Let G be a monoid acting on a topological space X . Prove the following statements:

- (1) If X is G -noetherian, and $Y \subset X$ is a G -invariant subspace with the induced topology, then Y is also G -noetherian.
- (2) If $Y, Y' \subset X$ are G -invariant subspaces and are both G -noetherian, then so is $Y \cup Y'$.
- (3) If X' is another space with a G -action and $f: X \rightarrow X'$ is G -equivariant and continuous, then the image of f is G -noetherian.
- (4) Suppose G is a group and let $G' \subseteq G$ be a subgroup. Let G' act on $G \times X$ by $h \cdot (g, x) = (gh^{-1}, hx)$ for $h \in G'$. If X is G' -noetherian, then the space $(G \times X)/G'$ is G -noetherian. \square

Our goal is the following:

Theorem 6.4.8 (Draisma–Kuttler). *The space $Y^{\leq r}$ is G_∞ -noetherian.*

Proof. This is proven by induction on r . The case $r = 0$ is trivial since $Y^{\leq 0}$ is a single point.

Now assume that $r > 0$ and that $Y^{\leq (r-1)}$ is G_∞ -noetherian. By Theorem 6.4.5, we can find finitely many pairs $\mathbf{w}_1, \mathbf{w}'_1, \dots, \mathbf{w}_N, \mathbf{w}'_N$ such that

- (1) Each \mathbf{w}_i has length r . If we write it as (v_1, \dots, v_r) , then each v_i has finite support.
- (2) For each i , the components of \mathbf{w}_i and \mathbf{w}'_i have disjoint support.
- (3) $Y^{\leq (r-1)}$ is set-theoretically defined by the G_∞ -orbits of $\det(M[\mathbf{w}_i; \mathbf{w}'_i])$.

For each i , define

$$Z_i = Y^{\leq r} \setminus V(G_\infty \det M[\mathbf{w}_i; \mathbf{w}'_i]).$$

Then Z_i is an open subset of $Y^{\leq r}$ which is closed under G_∞ , and we have

$$Y^{\leq r} = Y^{\leq (r-1)} \cup Z_1 \cup Z_2 \cup \cdots \cup Z_N.$$

By Exercise 6.4.7(2), it will suffice to show that each Z_i is G_∞ -noetherian. So fix one such i and set $Z = Z_i$, $\mathbf{w}_i = (w_1, \dots, w_r)$, $\mathbf{w}'_i = (w'_1, \dots, w'_r)$. Let p be the maximum of the union of the supports of all w_i and w'_i .

Recall that we have a copy of $\Sigma_\infty \subset G_\infty$ which permutes tensor factors. Let $\Sigma \subset \Sigma_\infty$ be the subgroup of σ such that $\sigma(i) = i$ for $i = 1, \dots, p$. Define

$$Z' = Y^{\leq r} \setminus V(\det M[\mathbf{w}; \mathbf{w}']).$$

Then Z' is invariant under Σ . By the next lemma, Z' is Σ -noetherian. Now define a map

$$\begin{aligned} (G_\infty \times Z')/\Sigma &\rightarrow Z \\ (g, z) &\mapsto g \cdot z. \end{aligned}$$

This is both G_∞ -equivariant and surjective, so by Exercise 6.4.7, Z is G_∞ -noetherian, and we are done. \square

Lemma 6.4.9. *Z' is Σ -noetherian.*

Proof. We continue to use the same notation from the previous proof.

Let $\Gamma \subset \text{Inc}(\mathbf{Z}_{>0})$ be the submonoid of increasing functions which are the identity on $1, \dots, p$. Note that Γ acts on $Y^{\leq r}$: given an increasing function f and a vector $v \in Y^{\leq r}$, there are only finitely many basis vectors e_I needed to write v , and hence the union of the I have finite support, let q be the maximal value. Then for any two permutations σ, σ' whose inverses agree on $1, \dots, q$, we have $\sigma(v) = \sigma'(v)$. In particular, pick any $\sigma \in \Sigma$ whose inverse agrees with f on $1, \dots, q$ and define $f(v) = \sigma(v)$. It will suffice to show that Z' is Γ -noetherian, since the Γ -orbit of an equation is contained in its Σ -orbit.

Let J be the set of infinite words w in $\{0, \dots, r\}$ such that the support of w contains at most 1 element in positions beyond p . Let \mathbf{k}^J be the vector space with basis indexed by J . Setting $R = \mathbf{k}[x_w \mid w \in J]$, \mathbf{k}^J is a subset of $\text{Spec}(R)$. By setting the noetherian ring in Theorem 6.3.2 to be the polynomial ring over \mathbf{k} in the variables x_w where w ranges over the (finitely many) words with support contained in $[p]$, we deduce that R is Γ -noetherian, and hence the same is true for \mathbf{k}^J .

Then $\det M[\mathbf{w}; \mathbf{w}']$ can be interpreted as a function on \mathbf{k}^J ; let Q be the open subset where this function is nonzero. We define a function $\pi: Z' \rightarrow Q$ which sends a point to the values of its coordinates x_w where $w \in J$. This is continuous and Γ -equivariant. In fact, it is an embedding. We omit the details and refer the reader to [Dr2, §7], but the point is that on the subspace Z' , one can rewrite any coordinate $x_{w'}$ in terms of the coordinates x_w with $w \in J$. Assuming this embedding property, we see that Z' inherits Γ -noetherianity from Q , so we are done. \square

Remark 6.4.10. By Exercise 6.4.2, $Y^{\leq 1} = X^{\leq 1}$. So $Y^{\leq 1}$ consists of two G_∞ -orbits: the points $\{0\}$ and the set of nonzero rank 1 tensors. So the theorem is also trivial in this case. \square

Lemma 6.4.11. *If $Z \subset Y^{\leq r}$ is a closed G_∞ -subset, then it is the common solution set of finitely many G_∞ -orbits of polynomials. In particular, it is defined by bounded degree polynomials.*

Proof. If not, we can find an infinite sequence of polynomials f_1, f_2, \dots that all vanish on Z , but such that Z_i , the common solution set of $G_\infty f_1, \dots, G_\infty f_i$, satisfies $Z_1 \supsetneq Z_2 \supsetneq \dots$ in direct contradiction to (2). \square

Taking $Z = X^{\leq r}$, we get that it is defined by bounded degree polynomials.

6.4.4. *Finishing the proof.* To finish, pick $e_0 \in V$ such that $x_0(e_0) = 1$. Define

$$\begin{aligned} \tau: V^{\otimes p} &\rightarrow V^{\otimes(p+1)} \\ \omega &\mapsto \omega \otimes e_0. \end{aligned}$$

Given $\omega \in V^{\otimes p}$, we can define $\omega_\infty = \omega \otimes e_0^{\otimes \infty} \in V^{\otimes \infty}$, and their ranks are the same. So pulling back all of the equations vanishing on $X^{\leq r}$ gives equations that define the border rank $\leq r$ locus in $V^{\otimes p}$.

6.5. **Variants.** A similar situation can be considered with both exterior and symmetric powers of vector spaces instead of tensor powers. Given V , an element of $\bigwedge^n V$ of the form $v_1 \wedge \cdots \wedge v_n$ is said to have rank 1. Rank and border rank are defined similarly. For $\text{Sym}^n V$, elements of the form v^n are said to have rank 1 (technically we should be working with the divided power instead of the symmetric power). We note some theorems:

Theorem 6.5.1 (Draisma–Eggermont [DE]). *For each r , there is a constant $C(r)$ such that the variety of border rank $\leq r$ anti-symmetric tensors is cut out by polynomials of degree $\leq C(r)$. The constant is independent of n and also V .*

Theorem 6.5.2 (Sam [Sa]). *Fix a field of characteristic 0. For each r , there is a constant $C(r)$ such that the full ideal of polynomials vanishing on the variety of border rank $\leq r$ symmetric tensors is generated by polynomials of degree $\leq C(r)$. The constant is independent of n and also V .*

The last theorem gives an ideal-theoretic statement which improves the set-theoretic statement (though is limited by its restriction on characteristic).

7. MORE ON FI-MODULES

7.1. **Asymptotic combinatorial properties.** In a homework problem, you were asked to show that given a finitely generated **FI**-module M over a field \mathbf{k} , the function $n \mapsto \dim_{\mathbf{k}} M_n$ is a polynomial for $n \gg 0$. Each M_n is a representation of the symmetric group Σ_n . If \mathbf{k} is a field of characteristic 0, then M_n decomposes into a direct sum of irreducible representations, which are indexed by partitions λ of n , say that \mathbf{M}_λ appears with multiplicity $m_{M,\lambda}$. We'd like to understand how these multiplicities behave.

Recall that the principal projective module P_n satisfies $P_n(m) = \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} \mathbf{k}[\Sigma_n]$ where Σ_{m-n} acts trivially on $\mathbf{k}[\Sigma_n]$. We have an action of Σ_n on P_n by pre-composition, which commutes with post-composition by injections (this is just the associativity of composing functions). In that sense, we can decompose P_n into a direct sum

$$P_n = \bigoplus_{\lambda \vdash n} P_\lambda^{\oplus \dim \mathbf{M}_\lambda}$$

which follows from the decomposition of the group algebra

$$\mathbf{k}[\Sigma_n] = \bigoplus_{\lambda \vdash n} \mathbf{M}_\lambda^{\oplus \dim \mathbf{M}_\lambda}.$$

In particular, we have

$$P_\lambda(m) = \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} \mathbf{M}_\lambda = \bigoplus_{\nu \vdash m} \mathbf{M}_\nu$$

where, by Pieri's rule (Theorem 2.7.3), the last sum is over all ν containing λ such that ν/λ is a horizontal strip of size $m - n$. The submodule structure of P_λ can be determined completely from the following result whose proof we will omit.

Proposition 7.1.1. *Given ν such that ν/λ is a horizontal strip, the submodule of P_λ generated by \mathbf{M}_ν contains all $\mathbf{M}_{\nu'}$ such that $\nu \subseteq \nu'$.*

Recall that P_n satisfies the universal property that a map of **FI**-modules $P_n \rightarrow M$ is the same as an Σ_n -equivariant map $\mathbf{k}[\Sigma_n] \rightarrow M(n)$, which is again equivalent to a choice of arbitrary element in $M(n)$. Similarly, a map of **FI**-modules $P_\lambda \rightarrow M$ is the same as an Σ_n -equivariant map $\mathbf{M}_\lambda \rightarrow M(n)$.

To add a horizontal strip to λ , we add some subset of boxes to the first λ_1 columns, and the rest must go in the first row. Let μ^1, \dots, μ^c be all possible partitions that can be obtained by adding a horizontal strip to λ by only adding boxes in the first λ_1 columns. Then we see that every partition in the decomposition of $P_\lambda(m)$ must be of the form $\mu^i + (m - |\mu^i|)$ for some i and where addition for partitions is defined componentwise.

Example 7.1.2. If $\lambda = (3, 1)$, then $c = 6$ with $\mu^1 = \lambda = (3, 1)$, $\mu^2 = (3, 1, 1)$, $\mu^3 = (3, 2)$, $\mu^4 = (3, 2, 1)$, $\mu^5 = (3, 3)$, and $\mu^6 = (3, 3, 1)$. If we add a horizontal strip to λ to get a partition of size m , then it must be one of

$$(m-1, 1), \quad (m-2, 1, 1), \quad (m-2, 2), \quad (m-3, 2, 1), \quad (m-3, 3), \quad (m-4, 3, 1). \quad \square$$

Let μ be the largest of the μ^i , i.e., it is obtained by adding all possible boxes in the first λ_1 columns to λ (in the previous example, it is $(3, 2, 1)$). Then we have a map $P_\mu \rightarrow P_\lambda$ by the above discussion by taking the identity map $\mathbf{M}_\mu \rightarrow \mathbf{M}_\mu \subset P_\lambda(|\mu| - |\lambda|)$. If \mathbf{M}_ν is in the image of this map, then both ν/λ and ν/μ must be horizontal strips, and this is only possible for partitions of the form $\mu + (k)$. In fact, by Proposition 7.1.1, $\mathbf{M}_{\mu+(k)}$ is in the image for all $k \geq 0$. In any case, one can argue that the sum of the subspaces $\mathbf{M}_{\mu+(k)}$ is an **FI**-submodule of P_λ , which we will temporarily denote by V_μ^+ .

Proposition 7.1.3. *V_μ^+ is a finitely generated **FI**-module.*

Proof. V_μ^+ is a submodule of P_λ , which is finitely generated, so the result follows from noetherianity of **FI**-modules. \square

Proposition 7.1.4. *Let M be a finitely generated **FI**-module over a field of characteristic 0. For a partition λ , let $m_{M,\lambda}$ be the multiplicity of \mathbf{M}_λ in $M(|\lambda|)$. Then, for any λ , the function $i \mapsto m_{M,\lambda+(i)}$ is constant for $i \gg 0$.*

Proof. By Exercise 5.1.2, the Segre product of two finitely generated **FI**-modules is again finitely generated. In particular, we have a finitely generated **FI**-module of the form $[n] \mapsto M(n) \otimes_{\mathbf{k}} V_\lambda^+(n) = M(n) \otimes_{\mathbf{k}} \mathbf{M}_{\lambda+(n-|\lambda|)}$. Note that

$$\dim_{\mathbf{k}}(M(n) \otimes_{\mathbf{k}} \mathbf{M}_{\lambda+(n-|\lambda|)})^{\Sigma_n} = \dim_{\mathbf{k}}(M(n) \otimes_{\mathbf{k}} \mathbf{M}_{\lambda+(n-|\lambda|)})_{\Sigma_n} = m_{M,\lambda+(n-|\lambda|)}$$

where the first equality follows from semisimplicity, and the second follows from self-duality of representations of Σ_n . In particular, Proposition 5.2.5 implies that there is a finitely generated graded $\mathbf{k}[t]$ -module whose degree i piece has dimension $m_{M,\lambda+(i)}$, so this value is constant for $i \gg 0$. \square

Recall that in §2.12, we defined the Kronecker coefficients by

$$g_{\lambda,\mu,\nu} = \dim_{\mathbb{C}}(\mathbf{M}_{\lambda} \otimes \mathbf{M}_{\mu} \otimes \mathbf{M}_{\nu})^{\Sigma_n},$$

where λ, μ, ν are partitions of n . Alternatively, since Σ_n -representations are self-dual, this is the multiplicity of \mathbf{M}_{λ} in $\mathbf{M}_{\mu} \otimes \mathbf{M}_{\nu}$.

Corollary 7.1.5 (Murnaghan). *Let λ, μ, ν be partitions of the same size. Then the function*

$$i \mapsto g_{\lambda+(i),\mu+(i),\nu+(i)}$$

is constant for $i \gg 0$.

Proof. In the above proposition, take M to be the Segre product of V_{μ}^{+} and V_{ν}^{+} . □

The value

$$\bar{g}_{\lambda,\mu,\nu} = \lim_{i \rightarrow \infty} g_{\lambda+(i),\mu+(i),\nu+(i)}$$

is the **stable Kronecker coefficient**, and are in some sense, supposed to simpler than the usual Kronecker coefficients. Our indexing is different from the standard indexing in the literature: there one often replaces λ, μ, ν by the partitions obtained by removing the first part.

Example 7.1.6. • When $\lambda = \mu = \nu = \emptyset$, then $g_{i,i,i} = 1$ for all i since \mathbf{M}_i is the trivial representation of Σ_i .

- When $\lambda = \mu = \nu = (1, 1)$, then $g_{(i+1,1),(i+1,1),(i+1,1)}$ is 1 if $i > 0$ and is 0 for $i = 0$. The case $i = 0$ follows from the fact that $M_{(1,1)}$ is the sign representation of Σ_2 . In general, for partitions of size n , $g_{\alpha,\beta,(n-1,1)} + 1$ is the number of ways to remove a box from α and then add a box to get β . □

More generally, one can add other sequences to λ, μ, ν . For example, fix 3 other partitions α, β, γ of the same size and consider the sequence

$$i \mapsto g_{\lambda+i\alpha,\mu+i\beta,\nu+i\gamma}.$$

What kind of behavior does this exhibit? In fact, one has the following result (see [Ste, SS2]):

Theorem 7.1.7 (Stembridge, Sam–Snowden). *If $g_{i\alpha,i\beta,i\gamma} = 1$ for all i , then the sequence $i \mapsto g_{\lambda+i\alpha,\mu+i\beta,\nu+i\gamma}$ is constant for $i \gg 0$. Conversely, if this sequence is eventually constant for all λ, μ, ν , then $g_{i\alpha,i\beta,i\gamma} = 1$ for all i .*

Murnaghan’s theorem is the special case when $\alpha = \beta = \gamma = (1)$ (cf. Example 7.1.6).

The techniques used here will not suffice to prove this generalization, which uses a little bit of algebraic geometry, see [SS2] for more details.

7.2. Serre quotient categories. To understand the role of the V_{λ}^{+} in the asymptotic behavior of finitely generated **FI**-modules, we use the notion of a Serre quotient of an abelian category.

Definition 7.2.1. Let \mathcal{C} be an abelian category. A **Serre subcategory** \mathcal{C}' of \mathcal{C} is a full subcategory (i.e., $\text{Hom}_{\mathcal{C}'}(x, y) = \text{Hom}_{\mathcal{C}}(x, y)$ for all $x, y \in \mathcal{C}'$) such that given a short exact sequence of objects in \mathcal{C}

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0,$$

we have $B \in \mathcal{C}'$ if and only if $A, C \in \mathcal{C}'$. □

This comes up as follows. Suppose we are given an exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ between abelian categories. Then the full subcategory on the objects $\{x \in \mathcal{C} \mid F(x) = 0\}$ is a Serre subcategory. In this case, we can think of this subcategory as being the kernel of the functor F . Then, mimicking more familiar constructions in algebra, we will also want to define the quotient by a Serre subcategory.

Definition 7.2.2. Let $\mathcal{C}' \subset \mathcal{C}$ be a Serre subcategory and let $x, y \in \mathcal{C}$ be objects. Consider the category $\mathcal{D}(x, y)$ whose objects are pairs (x', y') where $x' \subseteq x$ is a subobject such that $x/x' \in \mathcal{C}'$ and $y' \subseteq y$ is a subobject such that $y' \in \mathcal{C}'$. We have a morphism $(x', y') \rightarrow (x'', y'')$ if $x'' \subseteq x'$ and $y' \subseteq y''$. The assignment $(x', y') \mapsto \text{Hom}_{\mathcal{C}'}(x', y/y')$ is a functor on the category $\mathcal{D}(x, y)$.

The **quotient category** \mathcal{C}/\mathcal{C}' has the same objects as \mathcal{C} , and the hom set between x and y is defined by

$$\text{Hom}_{\mathcal{C}/\mathcal{C}'}(x, y) = \varinjlim_{(x', y') \in \mathcal{D}(x, y)} \text{Hom}_{\mathcal{C}'}(x', y/y'). \quad \square$$

Recall that the colimit can be defined as the direct sum $\bigoplus_{(x', y') \in \mathcal{D}(x, y)} \text{Hom}_{\mathcal{C}'}(x', y/y')$ modulo the equivalence relation that identifies a morphism $x' \rightarrow y/y'$ with the corresponding morphism $x'' \rightarrow y/y''$ if we have a morphism $(x', y') \rightarrow (x'', y'')$ in $\mathcal{D}(x, y)$.

To compose maps $x \rightarrow y$ and $y \rightarrow z$ in \mathcal{C}/\mathcal{C}' , first pick representatives $f: x' \rightarrow y/y'$ and $g: y'' \rightarrow z/z'$. Let $x'' \subseteq x'$ be the preimage of y''/y' under f and let z'' be the sum of z' and the image of g . Then we can restrict f and g to get representatives $f': x'' \rightarrow y/y'$ and $g': y'' \rightarrow z/z''$, which are actually composable since the image of the first map is y'' . The composition $x \rightarrow y \rightarrow z$ is then represented by $g' \circ f'$.

In more detail, a morphism $x \rightarrow y$ in the quotient category can be represented by a morphism $x' \rightarrow y/y'$ where x/x' and y' are both in \mathcal{C}' . Given a subobject $y'' \subseteq y'$, we get a natural map $\text{Hom}_{\mathcal{C}}(x', y/y'') \rightarrow \text{Hom}_{\mathcal{C}}(x', y/y')$, and any morphism in the source represents the same morphism as its image. Similarly, given $x' \subseteq x'' \subseteq x$, x/x'' is a quotient of x/x' and hence belongs to \mathcal{C}' , and we get a natural map $\text{Hom}_{\mathcal{C}}(x'', y/y') \rightarrow \text{Hom}_{\mathcal{C}}(x', y/y')$ and any morphism in the source represents the same morphism as its image.

In particular, x and y are isomorphic in \mathcal{C}/\mathcal{C}' if and only if there are maps $x \xleftarrow{f} z \xrightarrow{g} y$ such that the kernel and cokernel of both f, g are in \mathcal{C}' .

In fact, \mathcal{C}/\mathcal{C}' is an abelian category, and the quotient functor $T: \mathcal{C} \rightarrow \mathcal{C}/\mathcal{C}'$ is exact, and we call it the **localization functor**. It satisfies the following universal property: any exact functor $F: \mathcal{C} \rightarrow \mathcal{D}$ such that $F(x) = 0$ for all $x \in \mathcal{C}'$ factors through T , i.e., there is a functor $F': \mathcal{C}/\mathcal{C}' \rightarrow \mathcal{D}$ such that $F = F' \circ T$.

Example 7.2.3. Let \mathcal{C} be the category of finitely generated R -modules for a PID R . Let \mathcal{C}' be the full subcategory on the torsion R -modules. Then \mathcal{C}/\mathcal{C}' is equivalent to the category of finite-dimensional K -vector spaces, where K is the fraction field of R . \square

Example 7.2.4 (For those who know some algebraic geometry). Let \mathbf{k} be a field and let \mathcal{C} be the category of finitely generated graded $\mathbf{k}[x_1, \dots, x_n]$ -modules (here $\deg x_i = 1$ for all i) and let \mathcal{C}' be the full subcategory on the torsion modules. In this case, \mathcal{C}/\mathcal{C}' is equivalent to the category of coherent sheaves on projective space $\mathbf{P}_{\mathbf{k}}^{n-1}$. \square

Inspired by these examples, an important case to consider is when \mathcal{C} is the category of modules of some kind, and \mathcal{C}' is the subcategory of “torsion” modules. Then \mathcal{C}/\mathcal{C}' can be thought of as the category of modules over the “fraction field”, even if such an object does

not exist. Borrowing terminology from algebraic geometry, where the fraction field of an integral domain corresponds to the “generic point” of the corresponding affine scheme, we call \mathcal{C}/\mathcal{C}' the **generic category**.

7.3. Generic FI-modules. An element $x \in M(n)$ of an **FI**-module M is **torsion** if there exists an integer m and a non-invertible morphism $f: [n] \rightarrow [m]$ such that $f_*(x) = 0$. Since Σ_m acts transitively on all morphisms $[n] \rightarrow [m]$, this implies that $f_*(x) = 0$ for all non-invertible morphisms $f: [n] \rightarrow [m]$. In particular, the set of torsion elements of an **FI**-module is a submodule, called the **torsion submodule**, and denoted M^{tors} . An **FI**-module M is **torsion** if $M = M^{\text{tors}}$.

Lemma 7.3.1. *If M is finitely generated and torsion, then $M(m) = 0$ for $m \gg 0$.*

Proof. Say that M is generated by x_1, \dots, x_r , where $x_i \in M(n_i)$. Since each is torsion, there exists m_i such that $f_*(x_i) = 0$ for all $f: [n_i] \rightarrow [m]$ with $m \geq m_i$. In particular, $M(m) = 0$ for $m \geq \max(m_1, \dots, m_r)$. \square

We consider the above situation with \mathcal{C} being the category of finitely generated **FI**-modules, and \mathcal{C}' being the full subcategory of torsion modules (it is easy to verify that \mathcal{C}' is a Serre subcategory).

Given an integer n and an **FI**-module M , let $M^{\geq n}$ be the **FI**-submodule of M generated by all elements of degrees $\geq n$. Call it the degree n truncation of M . In particular,

$$M^{\geq n}(m) = \begin{cases} M(m) & \text{if } m \geq n \\ 0 & \text{else} \end{cases}.$$

Proposition 7.3.2. *Two finitely generated **FI**-modules M and N are isomorphic in the generic category \mathcal{C}/\mathcal{C}' if and only if $M^{\geq n} \cong N^{\geq n}$ for some n .*

Proof. Note that $M/M^{\geq n}$ and $N/N^{\geq n}$ are both objects in \mathcal{C}' , so if $M^{\geq n} \cong N^{\geq n}$, pick an isomorphism φ . Then φ is an element in the colimit defining $\text{Hom}_{\mathcal{C}/\mathcal{C}'}(M, N)$, and its inverse φ^{-1} is also an element in the colimit defining $\text{Hom}_{\mathcal{C}/\mathcal{C}'}(N, M)$, and their compositions in both directions represent the identity.

Conversely, if M and N are isomorphic in \mathcal{C}/\mathcal{C}' , then pick an isomorphism φ with inverse φ^{-1} . Then there exists $M' \subseteq M$ and $N' \subseteq N$ such that M/M' and N' are torsion, and an **FI**-module morphism $f: M' \rightarrow N/N'$ which represents φ in the colimit. Similarly, there exists $N'' \subseteq N$ and $M'' \subseteq M$ such that N/N'' and M'' are torsion and an **FI**-module morphism $g: N'' \rightarrow M/M''$ which represents φ^{-1} in the colimit. Up to the equivalence relation on the colimit, their composition is defined on some pair (M''', N''') such that M/M''' and N''' are torsion. By Lemma 7.3.1, these are 0 in large enough degrees, say larger than n . The restrictions of f and g to the degree n truncations are then inverses of one another. \square

A particular case of truncations: if $|\lambda| = k$, then $(V_\lambda^+)^{\geq n} = V_{\lambda+(n-k)}^+$, as follows from Proposition 7.1.1. In particular, these truncations are all isomorphic to each other in the generic category \mathcal{C}/\mathcal{C}' . In fact, since the size of the first part is irrelevant, let $\mu = (\lambda_2, \lambda_3, \dots)$, and call the resulting object L_μ in \mathcal{C}/\mathcal{C}' .

In fact, L_μ is a simple object: any nonzero subobject would be represented by a subobject $V_{\lambda+(n+k)}^+$ in V_λ , and the quotient is torsion, so the only nonzero subobject of L_μ is L_μ itself. We can be more precise about the structure of the generic category (proof omitted):

Theorem 7.3.3. (1) *Every simple object of \mathcal{C}/\mathcal{C}' is isomorphic to L_μ for some μ , and they are all mutually non-isomorphic.*
(2) *Every object of \mathcal{C}/\mathcal{C}' has a finite composition series.*

This structural result implies that many asymptotic properties about finitely generated **FI**-modules can be deduced from the corresponding properties of the V_λ^+ , which involve “growing the first row to infinity”.

Remark 7.3.4. Given an abelian category \mathcal{C} , say that an object is dimension 0 if every finitely generated subobject has a finite composition series. Let \mathcal{C}' be the full subcategory of dimension 0 objects. By induction, say that M has dimension d if $T(M) \in \mathcal{C}/\mathcal{C}'$ has dimension $d - 1$. For any object where this definition is not applicable, i.e., it does not become dimension 0 after any finite number of iterations of quotienting by dimension 0 objects, then define its dimension to be ∞ .

The **Gabriel–Krull dimension** of \mathcal{C} is defined to be the supremum of the dimension of all objects. Our discussion above says that the category of **FI**-modules has dimension 1. This is some measure of complexity of the category: when R is a noetherian commutative ring, the Gabriel–Krull dimension of its module category agrees with its Krull dimension [**★ Steven: double-check assumptions ★**]. \square

7.4. Semi-induced FI-modules. The projective **FI**-modules P_λ in characteristic 0 are “induced modules” in the sense that we have

$$P_\lambda(m) \cong \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} \mathbf{M}_\lambda$$

for all $m \geq n$, and where $n = |\lambda|$.

Now consider **FI**-modules over a general field \mathbf{k} . Let V be a (finite-dimensional) representation of Σ_n for some n . If x_1, \dots, x_r generate V as a Σ_n -module, then we have a surjection $\mathbf{k}[\Sigma_n]^{\oplus r} \rightarrow V$ via $(\alpha_1, \dots, \alpha_r) \mapsto \sum_i \alpha_i x_i$. Let K be the kernel. Then for each $m \geq n$, we have a short exact sequence

$$0 \rightarrow \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} K \rightarrow \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} \mathbf{k}[\Sigma_n]^{\oplus r} \rightarrow \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} V \rightarrow 0.$$

The middle term is $P_n^{\oplus r}(m)$, and since the Σ_n action on P_n commutes with the **FI**-module structure, we have a submodule $\mathcal{J}(K) \subset P_n^{\oplus r}$ given by

$$\mathcal{J}(K)(m) = \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} K,$$

and a quotient module $P_n^{\oplus r} \rightarrow \mathcal{J}(V)$ given by

$$\mathcal{J}(V)(m) = \text{Ind}_{\Sigma_n \times \Sigma_{m-n}}^{\Sigma_m} V.$$

Since V was arbitrary, the construction of $\mathcal{J}(V)$ makes sense in general, though it may not be clear that it is independent of the choice of generators. Alternatively, one can use the twisted commutative algebra perspective and define V as $A \otimes V$ where A is the free twisted commutative algebra generated by a single element of degree 1 (see §§3.1 and 3.2).

In any case, we call $\mathcal{J}(V)$ an **induced FI-module**.

Exercise 7.4.1. Show that $\mathcal{J}(V)$ is a projective **FI**-module if and only if V is a projective $\mathbf{k}[\Sigma_n]$ -module (which is automatic if $\mathbf{k}[\Sigma_n]$ is semisimple, for example in characteristic 0). \square

Remark 7.4.2. In the language of relative homological algebra [Ho], $\mathcal{J}(V)$ is a relatively projective module with respect to the pair of “rings” $(\mathbf{k}[\mathbf{FI}], \mathbf{k}[\Sigma_n])$. To define this in general, consider a ring R with a subring S . Then an exact sequence of R -modules is (R, S) -**exact** if

the kernel of each map is a direct summand of the source as an S -module. Then an R -module P is (R, S) -**projective** if, for every (R, S) -exact sequence

$$0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$$

and every R -linear map $h: P \rightarrow M_3$, there is a lifting of this map $h': P \rightarrow M_2$ meaning that $h = g \circ h'$. \square

An **FI**-module M is **semi-induced** if it has a finite filtration $M = M_r \supset M_{r-1} \supset \cdots \supset M_1 \supset M_0 = 0$ such that M_i/M_{i-1} is an induced **FI**-module for $i = 1, \dots, r$. Consider the functor $\mathbf{FI} \rightarrow \mathbf{FI}$ which sends a finite set S to the disjoint union with a new element $S \amalg \{*\}$ and an injection $f: S \rightarrow T$ to the corresponding injection $f': S \amalg \{*\} \rightarrow T \amalg \{*\}$ which is f on S and sends $*$ to $*$. Given an **FI**-module M , we let ΣM be the pullback of M along this functor. Explicitly, this means that $(\Sigma M)(S) = M(S \amalg \{*\})$ and that the action of f comes from the action of f' . Then ΣM is called the **shift** of M since $(\Sigma M)(n) = M(n+1)$.

There is a canonical map $M \rightarrow \Sigma M$ where the map $M(S) \rightarrow (\Sigma M)(S)$ comes from the inclusion $S \rightarrow S \amalg \{*\}$. By composing, we also get a canonical map $M \rightarrow \Sigma^n M$ for each n .

Exercise 7.4.3. Show that the kernel of $M \rightarrow \Sigma^n M$ is a torsion module. \square

Let ΔM be the cokernel of the canonical map $M \rightarrow \Sigma M$ and let $h_M(n) = \dim_{\mathbf{k}} M(n)$. By the previous exercise and Lemma 7.3.1, the kernel of $M \rightarrow \Sigma M$ is 0 in sufficiently large degrees. In particular, for $n \gg 0$, we have

$$h_{\Delta M}(n) = h_{\Sigma M}(n) - h_M(n) = h_M(n+1) - h_M(n).$$

Exercise 7.4.4. If M is generated in degrees $\leq d$, show that ΔM is generated in degrees $\leq d-1$. By induction on d , and using the above equation, show that $h_M(n)$ agrees with a polynomial of degree $\leq d$ for $n \gg 0$. \square

The polynomial in the previous exercise will be denoted $p_M(n)$ and is called the **Hilbert polynomial** of M . Define the **degree** of M to be the degree of $p_M(n)$, and denote it $\delta(M)$. The degree of the 0 polynomial is -1 , by convention. So assuming $\delta(M) \geq 0$, we have

$$(7.4.5) \quad \delta(\Delta M) \leq \delta(M) - 1.$$

In fact, for the same reason, the same holds for the cokernel of the composition $M \rightarrow \Sigma M \rightarrow \Sigma^n M$ for any n .

We have the following important theorem about shifts and semi-induced modules [Na]:

Theorem 7.4.6 (Nagpal). *If M is a finitely generated **FI**-module, then $\Sigma^n M$ is semi-induced for $n \gg 0$.*

We omit the proof, but note the following corollary.

Corollary 7.4.7. *Let M be a finitely generated **FI**-module and set $k = \delta(M)$. There exists a complex of **FI**-modules*

$$0 \rightarrow M \rightarrow I^0 \rightarrow I^1 \rightarrow \cdots \rightarrow I^k \rightarrow 0$$

such that I^j is semi-induced for all j , and the homology of the complex is torsion.

Proof. We prove this by induction on k . If $k = -1$, then there is nothing to show since M is torsion and the complex $0 \rightarrow M \rightarrow 0$ satisfies the above conditions. In general for $k \geq 0$,

take n large enough so that $\Sigma^n M$ is semi-induced and set $I^0 = \Sigma^n M$. Then the cokernel N of the map $M \rightarrow \Sigma^n M$ satisfies $\delta(N) \leq \delta(M) - 1$ and so there exists a complex

$$0 \rightarrow N \rightarrow I^1 \rightarrow \cdots \rightarrow I^k \rightarrow 0$$

such that each I^j is semi-induced and the homology is torsion. Define $I^0 \rightarrow I^1$ to be the composition $I^0 \rightarrow N \rightarrow I^1$. Then the image of $I^0 \rightarrow I^1$ is the same as the image of $N \rightarrow I^1$ and the kernel is the preimage of the kernel of $N \rightarrow I^1$ under the surjection $I^0 \rightarrow N$. So the homology remains torsion and we have the desired complex. \square

7.5. Cohomology of FI-modules. In §4.3, we studied group homology with coefficients in a finitely generated **FI**-module M . In particular, in each degree, the direct sum $\bigoplus_n H_i(\Sigma_n; M(n))$ has the structure of a finitely generated $\mathbf{k}[t]$ -module. Now we investigate the situation with group cohomology, which will utilize Nagpal's theorem above.

Example 7.5.1. Let \mathbf{k} be a field of positive characteristic $p > 0$. Let \mathbf{k}^n be the standard permutation representation of Σ_n , and let V_n be the subspace of \mathbf{k}^n consisting of vectors whose coordinates sum to 0. If p divides n , then the line spanned by the all 1's vector is in V_n and hence $V_n^{\Sigma_n}$ is 1-dimensional. Otherwise, it is 0-dimensional. In other words,

$$\dim_{\mathbf{k}} H^0(\Sigma_n; V_n) = \begin{cases} 1 & \text{if } p \text{ divides } n \\ 0 & \text{else} \end{cases},$$

so the sequence of dimensions exhibits periodic behavior. Note that $[n] \mapsto \mathbf{k}^n$ is a finitely generated **FI**-module and that $[n] \mapsto V_n$ is a submodule of it. We will see more generally that periodic behavior occurs in the cohomology of finitely generated **FI**-modules. \square

Let M be an **FI**-module and, for each $i \geq 0$, define

$$\mathcal{H}^i(M) = \bigoplus_{n \geq 0} H^i(\Sigma_n; M(n)), \quad \mathcal{H}^*(M) = \bigoplus_{i \geq 0} \mathcal{H}^i(M).$$

Then $\mathcal{H}^*(M)$ is bigraded where $H^i(\Sigma_n; M(n))$ has bidegree (i, n) . Recall that in §3, we discussed the equivalence between the category of **FI**-modules and the category of modules over the twisted commutative algebra A which satisfies $A_n = \mathbf{k}$ for all n . To do this, we defined a monoidal structure on the category of symmetric sequences, which in one form becomes:

$$(V \otimes W)_n = \bigoplus_{i=0}^n \text{Ind}_{\Sigma_i \times \Sigma_{n-i}}^{\Sigma_n} (V_i \otimes W_{n-i}).$$

Given non-negatively bigraded vector spaces A and B (with finite-dimensional components), their tensor product is a bigraded vector space satisfying

$$(A \otimes B)_{i,j} = \bigoplus_{\substack{i'+i''=i \\ j'+j''=j}} A_{i',j'} \otimes B_{i'',j''}.$$

So we have a category of non-negatively bigraded vector spaces with this tensor product as its monoidal structure, and $\mathcal{H}^*(M)$ is an object of it.

Proposition 7.5.2. *The assignment $M \mapsto \mathcal{H}^*(M)$ is a monoidal functor, i.e., $\mathcal{H}^*(M \otimes N)$ is naturally isomorphic to $\mathcal{H}^*(M) \otimes \mathcal{H}^*(N)$.*

Proof. Fix (i, n) . Then we have

$$\begin{aligned}
 \mathcal{H}^*(M \otimes N)_{i,n} &= H^i(\Sigma_n; (M \otimes N)(n)) \\
 &= H^i(\Sigma_n; \bigoplus_{k=0}^n \text{Ind}_{\Sigma_k \times \Sigma_{n-k}}^{\Sigma_n} (M(k) \otimes N(n-k))) \\
 &= \bigoplus_{k=0}^n H^i(\Sigma_k \times \Sigma_{n-k}; M(k) \otimes N(n-k)) \\
 &= \bigoplus_{k=0}^n \bigoplus_{i'+i''=i} H^{i'}(\Sigma_k; M(k)) \otimes H^{i''}(\Sigma_{n-k}; N(n-k)) \\
 &= \bigoplus_{\substack{i'+i''=i \\ n'+n''=n}} \mathcal{H}^*(M)_{i',n'} \otimes \mathcal{H}^*(N)_{i'',n''},
 \end{aligned}$$

where the first two equalities are by definition, the third equality is Shapiro's lemma, and the fourth equality is the Künneth isomorphism. \square

In particular, since A is an algebra, applying \mathcal{H} to the multiplication map $A \otimes A \rightarrow A$ yields a map

$$\mathcal{H}(A) \otimes \mathcal{H}(A) \rightarrow \mathcal{H}(A).$$

By degree considerations, this restricts to a map $\mathcal{H}^0(A) \otimes \mathcal{H}^0(A) \rightarrow \mathcal{H}^0(A)$, and this is a commutative, associative algebra satisfying $\mathcal{H}^0(A) = \bigoplus_n H^0(\Sigma_n; \mathbf{k})$. Since we have a canonical identification $H^0(\Sigma_n; \mathbf{k}) = \mathbf{k}$, we let $x^{[n]}$ be the element corresponding to $1 \in \mathbf{k}$.

Lemma 7.5.3. *For all n, m , we have*

$$x^{[n]}x^{[m]} = \binom{n+m}{n} x^{[n+m]}.$$

Proof. Note that the map $H^0(\Sigma_n; \mathbf{k}) \otimes H^0(\Sigma_m; \mathbf{k}) \rightarrow H^0(\Sigma_{n+m}; \mathbf{k})$ is multiplication by some scalar, and we just have to show that it is $\binom{n+m}{m}$. In fact, this follows from the fact that the isomorphism coming from Shapiro's lemma can be described as a transfer map [Wei, Exercise 6.7.7]: in general, for a subgroup H of G and a representation M of G , the transfer map is induced by the map $M^H \rightarrow M^G$ given by $m \mapsto \sum gm$ where the sum is over a set of representatives of G/H . In our case, $M = \mathbf{k}$ is trivial and so this is multiplication by the index of $\Sigma_n \times \Sigma_m$ in Σ_{n+m} , which is $\binom{n+m}{n}$. \square

This algebra is the **divided power algebra** in 1 variable, and we will denote it \mathbf{D} . Note that if \mathbf{k} is a field of characteristic 0, then $\mathbf{D} \cong \mathbf{k}[t]$ under the identification $t^n/n! = x^{[n]}$. Over a field of positive characteristic, \mathbf{D} has many zerodivisors, and is not even finitely generated or noetherian.

Exercise 7.5.4. Let \mathbf{k} be a field of characteristic $p > 0$. Define a ring homomorphism $\mathbf{k}[y_0, y_1, \dots] \rightarrow \mathbf{D}$ by $y_i \mapsto x^{[p^i]}$. Show that this is surjective and that the kernel is the ideal $(y_0^p, y_1^p, y_2^p, \dots)$. \square

A commutative algebra R is **coherent** if every finitely generated ideal is a finitely presented R -module.

Exercise 7.5.5. Let R be a coherent ring and let M and N be finitely presented R -modules. Given a homomorphism $f: M \rightarrow N$, show that $\ker f$ and $\text{coker } f$ are finitely presented. \square

Proposition 7.5.6. \mathbf{D} is a coherent ring.

Proof. If \mathbf{k} is a field of characteristic 0, then $\mathbf{D} \cong \mathbf{k}[t]$, so it is noetherian, and hence coherent.

Otherwise, suppose \mathbf{k} is a field of positive characteristic $p > 0$. As in the previous exercise, identify \mathbf{D} with $\mathbf{k}[y_0, y_1, \dots]/(y_0^p, y_1^p, \dots)$. Any finite list of generators for an ideal only uses finitely many of the y_i , say the first r of them. Note that \mathbf{D} is a free module over the subring $R = \mathbf{k}[y_0, \dots, y_r]/(y_0^p, \dots, y_r^p)$, and so its presentation over \mathbf{D} can be obtained by taking a free presentation over R and then applying $- \otimes_R \mathbf{D}$. \square

Proposition 7.5.7. Suppose \mathbf{k} is a field of positive characteristic $p > 0$. Let M be a finitely presented graded \mathbf{D} -module. Then $n \mapsto \dim_{\mathbf{k}} M_n$ is a periodic function, whose period is a power of p , for $n \gg 0$.

Proof. As in the previous proof, identify \mathbf{D} with $\mathbf{k}[y_0, y_1, \dots]/(y_0^p, y_1^p, \dots)$. Pick a finite presentation of M , i.e., write it as the cokernel of a finite matrix with homogeneous entries from \mathbf{D} . It involves only finitely many of the y_i , say y_1, \dots, y_r , and let $R = \mathbf{k}[y_0, \dots, y_r]/(y_0^p, \dots, y_r^p)$. Let M' be the cokernel of the same matrix thought of now as a map of free modules over R .

Since R is finite-dimensional over \mathbf{k} , the same is true for M' . Furthermore, \mathbf{D} is a free module over R with basis given by the monomials in y_{r+1}, y_{r+2}, \dots with exponents between 0 and $p-1$. In particular, $M = M' \otimes_R \mathbf{D}$, and as a vector space, M is the tensor product of M' with the vector space whose basis is the set of monomials just listed. This implies that

$$\sum_{n \geq 0} (\dim_{\mathbf{k}} M_n) t^n = \frac{\sum_{n \geq 0} (\dim_{\mathbf{k}} M'_n) t^n}{1 - t^{p^{r+1}}}.$$

As just discussed, the numerator on the right hand side is a polynomial and hence the function $n \mapsto \dim_{\mathbf{k}} M_n$ is periodic of period p^{r+1} beyond the degree of the numerator. \square

The \mathbf{k} -linear map $d: \mathbf{D} \rightarrow \mathbf{D}$ defined by $d(x^{[n]}) = x^{[n-1]}$ is a **derivation**: for all $f, g \in \mathbf{D}$, we have $d(fg) = d(f)g + fd(g)$. Let M be a graded \mathbf{D} -module. A **connection** on M is a linear map $\nabla: M \rightarrow M$ that sends a homogeneous element of degree n to a homogeneous element of degree $n-1$, and satisfies $\nabla(fm) = d(f)m + f\nabla(m)$ for all $f \in \mathbf{D}$ and $m \in M$.

Proposition 7.5.8. Suppose M has a connection ∇ . Then the map

$$\begin{aligned} \psi: \mathbf{D} \otimes_{\mathbf{k}} \ker \nabla &\rightarrow M \\ \sum_i f_i \otimes m_i &\mapsto \sum_i f_i m_i \end{aligned}$$

is an isomorphism. In particular, M is a free \mathbf{D} -module.

Proof. First, we show that ψ is injective. If not, then we have $\sum_i f_i m_i = 0$ where the m_i are linearly independent elements of $\ker \nabla$. Furthermore, we may choose this so that the elements are homogeneous and so that $\deg(f_i) + \deg(m_i)$ is constant, and minimal amongst all relations that exist. In particular, $\deg(f_i) > 0$ for some i . Applying ∇ to this linear combination, we get

$$0 = \sum_i (d(f_i)m_i + f_i \nabla(m_i)) = \sum_i d(f_i)m_i.$$

This gives a linear combination of smaller degree, which contradicts our choice from before, so ψ is injective as claimed.

Now we show that ψ is surjective. Let $m \in M$ be an arbitrary element. Let r be the minimal integer such that $\nabla^r m = 0$ (r exists since ∇ is degree decreasing). By induction on r , we show that m is in the image of ψ . If $r = 0$, there is nothing to show, so suppose it holds for all values strictly smaller than r . Note that

$$m + \sum_{i=1}^{r-1} (-1)^i x^{[i]} \nabla^i(m) \in \ker \nabla.$$

By induction, each $\nabla^i(m)$ for $i > 0$ is in the image of ψ . Since every element of $\ker \nabla$ is in the image of ψ , we conclude that m is in the image of ψ . \square

By Proposition 7.5.2 and Lemma 7.5.3, for each $i \geq 0$, $\mathcal{H}^i(M) = \bigoplus_{n \geq 0} H^i(\Sigma_n; M(n))$ has the structure of a \mathbf{D} -module.

Proposition 7.5.9. *Let \mathbf{k} be a field of positive characteristic. The restriction map $H^i(\Sigma_n; \mathbf{k}) \rightarrow H^i(\Sigma_{n-1}; \mathbf{k})$ gives a connection on $\mathcal{H}^i(P_0)$. Furthermore, $\ker \nabla$ is 0 in degrees $> 2i$ and hence is finite-dimensional.*

Proof. We omit the proof of the first part, see [NS, §4.2] for the calculation. The second part is the dual version of Nakaoka’s stability theorem (Theorem 4.2.1) and the general fact that each of the (co)homology groups of a finite group with trivial coefficients is finite-dimensional (as follows from example, from the existence of the bar resolution [Wei, §6.5]). \square

Given a graded \mathbf{D} -module N and an integer n , we let $N^{\geq n}$ be the submodule of N generated by all elements of degree $\geq n$.

Theorem 7.5.10 (Nagpal–Snowden). *Let \mathbf{k} be a field of positive characteristic $p > 0$. Let M be a finitely generated \mathbf{FI} -module over \mathbf{k} .*

- (1) *There exists a finitely presented \mathbf{D} -module N and an integer m such that $N^{\geq m} \cong \mathcal{H}^i(M)^{\geq m}$.*
- (2) *The function $n \mapsto \dim_{\mathbf{k}} H^i(\Sigma_n; M(n))$ is a periodic function, whose period is a power of p , for $n \gg 0$.*

Proof. We reduce (1) to the case of semi-induced modules using Corollary 7.4.7, Exercise 7.5.5, and Proposition 7.5.6. Furthermore, using the long exact sequence on cohomology and doing induction on the length of the filtration for a semi-induced module whose quotients are induced, we can reduce to the case of an induced module $\mathcal{J}(V)$ with V a representation of Σ_n .

Recall that in the tca perspective, $\mathcal{J}(V)$ is the tensor product $A \otimes V$ where A is the tca freely generated by a single element of degree 1. The action of A is a map $A \otimes A \otimes V \rightarrow A \otimes V$, which is in fact the multiplication map on A tensored with the identity on V . By Proposition 7.5.2, after applying \mathcal{H}^* , the corresponding map $\mathcal{H}^*(A) \otimes \mathcal{H}^*(A) \otimes \mathcal{H}^*(V) \rightarrow \mathcal{H}^*(A) \otimes \mathcal{H}^*(V)$ is the tensor product of the multiplication map on $\mathcal{H}^*(A)$ with the identity on $\mathcal{H}^*(V)$. So $H^i(\mathcal{J}(V))$ is the \mathbf{D} -module $\bigoplus_{j=0}^i \mathcal{H}^{i-j}(A) \otimes H^j(\Sigma_n; V)$, and in particular, is a finitely generated free \mathbf{D} -module by Propositions 7.5.8 and 7.5.9.

- (2) follows from (1) and Proposition 7.5.7. \square

Finally, we return to the discussion of the cohomology of unordered configuration spaces of a topological space X in §5. Recall the definitions:

$$\begin{aligned}\mathrm{Conf}_n(X) &= \{(x_1, \dots, x_n) \in X^n \mid x_i \neq x_j \text{ for } i \neq j\}, \\ \mathrm{UConf}_n(X) &= \mathrm{Conf}_n(X)/\Sigma_n.\end{aligned}$$

Theorem 7.5.11. *Let \mathbf{k} be a field of positive characteristic p . Let X be a topological space with the property that the \mathbf{FI} -module $[n] \mapsto \mathrm{H}^i(\mathrm{Conf}_n(X); \mathbf{k})$ is finitely generated for all i . Then for each i , the function $n \mapsto \dim_{\mathbf{k}} \mathrm{H}^i(\mathrm{UConf}_n(X); \mathbf{k})$ is periodic, with period a power of p , for $n \gg 0$.*

Proof. Recall from our discussion on cohomology of configuration spaces that we have a Cartan–Leray spectral sequence (5.2.3):

$$E_2^{p,q} = \mathrm{H}^p(\Sigma_n; \mathrm{H}^q(\mathrm{Conf}_n(X); \mathbf{k})) \implies \mathrm{H}^{p+q}(\mathrm{UConf}_n(X); \mathbf{k}).$$

We take the direct sum of these spectral sequences over n . We focus on the case $p+q = i$. To compute the E_∞ page along this degree, we have to do computations with finitely many other degrees. By Theorem 7.5.10, each of the terms agrees with a finitely presented \mathbf{D} -module after some truncation. Since there are finitely many terms, we can find a single truncation that works for all of them. By Exercise 7.5.5 and Proposition 7.5.6, this implies that the E_∞ -terms in degree i agree with a finitely presented \mathbf{D} -module after some truncation. Finally, the term we want to compute, $\bigoplus_n \mathrm{H}^i(\mathrm{UConf}_n(X); \mathbf{k})$, is then built out of these terms using a finite number of extensions, so is itself a finitely presented \mathbf{D} -module after some truncation. Now we use Proposition 7.5.7. \square

8. Δ -MODULES

In §6, we studied the notion of tensor rank. Here we will come back to the rank 1 case from a different perspective and study its higher order syzygies.

8.1. Segre embeddings. First, we rephrase the setting. Given a vector space V , let $\mathbf{P}(V)$ denote the set of lines in V . Then $\mathbf{P}(V)$ is called the projectivization of V , and also projective space. Given a homogeneous polynomial $f \in \mathrm{Sym}(V^*)$, it does not make sense to ask for its value on a line $\ell \in \mathbf{P}(V)$, but it is well-defined whether $f(\ell) = 0$ or $f(\ell) \neq 0$. In particular, to any collection I of homogeneous polynomials, we can associate its vanishing locus $V(I)$. If we declare these to be closed subsets, then we get a topology on $\mathbf{P}(V)$, which is a graded variant of the Zariski topology on $\mathrm{Spec} R$ for a commutative ring R .

We focus on the **Segre embedding**:

$$\begin{aligned}\mathbf{P}(V_1) \times \cdots \times \mathbf{P}(V_n) &\rightarrow \mathbf{P}(V_1 \otimes \cdots \otimes V_n) \\ (\ell_1, \dots, \ell_n) &\mapsto \ell_1 \otimes \cdots \otimes \ell_n.\end{aligned}$$

Exercise 8.1.1. Check that the Segre embedding is in fact injective. \square

Note that if we take the union of the set of lines in the image of the Segre embedding, we get exactly the rank 1 tensors (and 0). So the image is a closed subset cut out by the same equations described in Remark 6.2.4.

Remark 8.1.2. The Segre embedding is not continuous if we put the product topology on the left hand side. Instead, we give the product of projective spaces the subspace topology coming from this embedding. \square

The Segre embeddings have a certain “factorization property”: for example, when $n = 3$, we can break it up into two steps:

$$\mathbf{P}(V_1) \times \mathbf{P}(V_2) \times \mathbf{P}(V_3) \rightarrow \mathbf{P}(V_1 \otimes V_2) \times \mathbf{P}(V_3) \rightarrow \mathbf{P}(V_1 \otimes V_2 \otimes V_3).$$

Using this, and a certain linearity property, all of the equations in Remark 6.2.4 can be built out of the simplest instance of a Segre embedding when $n = 2$ and $\dim V_1 = \dim V_2 = 2$.

Let’s consider that case in depth, but we’ll use dual spaces to make the functions not have a dual. Pick bases e_0, e_1 and f_0, f_1 for V_1^* and V_2^* and pick dual bases x_0, x_1 and y_0, y_1 so that $\text{Sym}(V_1) = \mathbf{k}[x_0, x_1]$ and $\text{Sym}(V_2) = \mathbf{k}[y_0, y_1]$. Let $z_{i,j} = x_i \otimes y_j$ so that $\text{Sym}(V_1 \otimes V_2) = \mathbf{k}[z_{0,0}, z_{0,1}, z_{1,0}, z_{1,1}]$. The image of the Segre embedding consists of all lines spanned by rank 1 matrices, so it is the solution set of the equation

$$\det \begin{pmatrix} z_{0,0} & z_{0,1} \\ z_{1,0} & z_{1,1} \end{pmatrix} = z_{0,0}z_{1,1} - z_{1,0}z_{0,1}.$$

The image in this case is also known as the quadric surface. As a representation of $\mathbf{GL}(V_1) \times \mathbf{GL}(V_2)$, the subspace generated by this equation in the space of degree 2 polynomials is

$$\bigwedge^2(V_1) \otimes \bigwedge^2(V_2) \subset \text{Sym}^2(V_1 \otimes V_2).$$

In fact, this implies that for arbitrary V_1, V_2 , the subspace $\bigwedge^2(V_1) \otimes \bigwedge^2(V_2)$ consists of equations that vanish on the image of the Segre embedding since it is true for every choice of 2-dimensional summands $V'_1 \subset V_1$ and $V'_2 \subset V_2$ and every element of $\bigwedge^2(V_1) \otimes \bigwedge^2(V_2)$ is a linear combination of elements living in summands of the form $\bigwedge^2(V'_1) \otimes \bigwedge^2(V'_2)$. Alternatively, we can argue that the space of degree 2-equations must be a polynomial subfunctor of $(V_1, V_2) \mapsto \text{Sym}^2(V_1 \otimes V_2)$ in the following sense.

We recall the notion of polynomial functor from §2.8.4 and generalize it at the same time. Let \mathbf{k} be an infinite field and let $\mathbf{Vec} = \mathbf{Vec}_{\mathbf{k}}$ be the category of finite-dimensional vector spaces. Let $\mathbf{Vec}_{\mathbf{k}}^n = \mathbf{Vec}^n = \mathbf{Vec}_{\mathbf{k}} \times \cdots \times \mathbf{Vec}_{\mathbf{k}}$, the n -fold product of this category, so objects are n -tuples of vector spaces and morphisms are n -tuples of linear maps. A functor $F: \mathbf{Vec}^n \rightarrow \mathbf{Vec}$ is a **polynomial functor** if, for every pair of n -tuples (V_1, \dots, V_n) and (W_1, \dots, W_n) , the map

$$\text{Hom}_{\mathbf{k}}(V_1, W_1) \times \cdots \times \text{Hom}_{\mathbf{k}}(V_n, W_n) \rightarrow \text{Hom}_{\mathbf{k}}(F(V_1, \dots, V_n), F(W_1, \dots, W_n))$$

is given by polynomial functions.

In particular, if we pick bases as above with the same convention, then all equations of the form $z_{i,j}z_{i',j'} - z_{i,j'}z_{i',j}$ vanish on the image.

Now let’s consider the factorization property. For simplicity, assume $n = 3$ and $\dim V_i = 2$ for all i . We let $z_{i,j}$ be the coordinate functions on $\mathbf{P}(V_1^* \otimes V_2^*)$ and let $z_{i,j,k}$ denote the coordinate functions on $\mathbf{P}(V_1^* \otimes V_2^* \otimes V_3^*)$. By repeating the above with the two vector spaces $V_1^* \otimes V_2^*$ and V_3^* , we conclude that the equations

$$z_{i,j,k}z_{i',j',k'} - z_{i,j,k'}z_{i',j',k}$$

vanish on the image of $\mathbf{P}(V_1^* \otimes V_2^*) \times \mathbf{P}(V_3^*)$. Since the triple product $\mathbf{P}(V_1^*) \times \mathbf{P}(V_2^*) \times \mathbf{P}(V_3^*)$ is in there, these equations also vanish there. However, we could have grouped together two of the vector spaces in two other ways (V_1^* with V_3^* , but also V_2^* with V_3^*). In particular, the

following two types of equations also vanish on the triple product:

$$\begin{aligned} z_{i,j,k} z_{i',j',k'} - z_{i,j',k} z_{i',j,k'} \\ z_{i,j,k} z_{i',j',k'} - z_{i',j,k} z_{i,j',k'}. \end{aligned}$$

By taking sums of these 3 types of equations, we can get all of the equations described in Remark 6.2.4. This argument extends to n -fold products of projective spaces for arbitrary n .

In particular, we see that the equation for the quadric surface “generates” all other quadratic equations for all Segre embeddings using this factorization property together with linearity. As we discussed, the ideals of the Segre embeddings are generated in degree 2, so this provides a complete description. Now we would like to axiomatize this structure and make precise in what sense this generation is happening.

8.2. Δ -modules. Define a category \mathbf{Vec}^Δ as follows. Its objects are finite collections of vector spaces $\{V_i\}_{i \in I}$ and a morphism $\{U_i\}_{i \in I} \rightarrow \{V_j\}_{j \in J}$ is a surjection $f: J \rightarrow I$ together with, for each $i \in I$, a linear map $\varepsilon_i: U_i \rightarrow \bigotimes_{j \in f^{-1}(i)} V_j$. Given another morphism $\{V_j\}_{j \in J} \rightarrow \{W_k\}_{k \in K}$ with surjection $g: K \rightarrow J$ and linear maps $\zeta_j: V_j \rightarrow \bigotimes_{k \in g^{-1}(j)} W_k$, the composition is given by the data $h: K \rightarrow I$ where $h = f \circ g$ and linear maps $\eta_i: U_i \rightarrow \bigotimes_{k \in h^{-1}(i)} W_k$ given by the composition of $\varepsilon_i: U_i \rightarrow \bigoplus_{j \in f^{-1}(i)} V_j$ with the tensor product of the maps $\zeta_j: V_j \rightarrow \bigotimes_{k \in g^{-1}(j)} W_k$ over all $j \in f^{-1}(i)$.

For each n , note that \mathbf{Vec}^n is a subcategory (not full) of \mathbf{Vec}^Δ consisting of the objects (V_1, \dots, V_n) and where the morphisms are the ones where $f: [n] \rightarrow [n]$ is the identity. Hence, given a functor $F: \mathbf{Vec}^\Delta \rightarrow \Delta$, we get a sequence of functors $F_n: \mathbf{Vec}^n \rightarrow \mathbf{Vec}$ by restricting to this subcategory \mathbf{Vec}^n . We will say that a functor $\mathbf{Vec}^\Delta \rightarrow \mathbf{Vec}$ is polynomial if each of the F_n is a polynomial functor in the sense described above. For short, polynomial functors $\mathbf{Vec}^\Delta \rightarrow \mathbf{Vec}$ are called **Δ -modules**.

Unfolding this definition, we see that a Δ -module is a sequence of polynomial functors $F_n: \mathbf{Vec}^n \rightarrow \mathbf{Vec}$ together with some extra data that relates them. Since they are polynomial functors, we get an action of $\prod_{i=1}^n \mathbf{GL}(V_i)$ on $F_n(V_1, \dots, V_n)$. Also, on the objects of the form (V_1, \dots, V_n) with all V_i equal to a single vector space V , there is an action of the symmetric group Σ_n on $F_n(V, \dots, V)$. The other key structure is that we have transition maps between the various F_n .

Example 8.2.1. The most basic example of a Δ -module is the functor

$$T: (V_i)_{i \in I} \mapsto \bigotimes_{i \in I} V_i.$$

To get other examples, we can compose this with a polynomial functor $\mathbf{Vec} \rightarrow \mathbf{Vec}$, such as symmetric and exterior powers (or even tensor products of such functors). So other examples of Δ -modules are,

$$\begin{aligned} (V_i)_{i \in I} \mapsto \mathrm{Sym}^r \left(\bigotimes_{i \in I} V_i \right), \\ (V_i)_{i \in I} \mapsto \bigwedge^r \left(\bigotimes_{i \in I} V_i \right), \end{aligned}$$

where r is some fixed nonnegative integer. We call them $\mathrm{Sym}^r(T)$ and $\bigwedge^r(T)$. Let $\mathrm{Sym}(T) = \bigoplus_{r \geq 0} \mathrm{Sym}^r(T)$ and $\bigwedge(T) = \bigoplus_{r \geq 0} \bigwedge^r(T)$. \square

An element of a Δ -module F is an element of some $F(\{V_i\}_{i \in I})$. Given any collection of elements, there is a smallest Δ -submodule containing it, which we call the submodule generated by these elements. We say that F is **finitely generated** if it can be generated by a finite collection of elements.

The basic functor T is finitely generated by a nonzero element in $F(\mathbf{k})$ where \mathbf{k} is the 1-dimensional vector space indexed by a singleton set.

Exercise 8.2.2. Show that the tensor power T^r defined by

$$T^r(\{V_i\}_{i \in I}) = \bigotimes_{i \in I} V_i^{\otimes r}$$

is a finitely generated Δ -module. In particular, finite tensor powers of symmetric and exterior powers (being quotients of T^r) are also finitely generated. \square

A key property is that the finite generation property is inherited by submodules (see [SS1, Theorem 9.2.3]):

Theorem 8.2.3. *Let F be a finitely generated Δ -module. Then F is noetherian, i.e., all Δ -submodules of F are again finitely generated.*

A key example of a submodule of $\text{Sym}^r(T)$ is to take the space of degree r equations which vanish on the Segre embedding of $\prod_{i \in I} \mathbf{P}(V_i^*)$. The previous section explains that for $r = 2$, this is generated by a single element. The general theorem says that this is finitely generated for any r .

As a final example, we note that one can define a Koszul complex:

$$\cdots \rightarrow \text{Sym}(T) \otimes \bigwedge^r(T) \rightarrow \cdots \rightarrow \text{Sym}(T) \otimes \bigwedge^2(T) \rightarrow \text{Sym}(T) \otimes T \rightarrow \text{Sym}(T).$$

This is the standard Koszul complex for $\text{Sym}(\bigotimes_{i \in I} V_i)$ when evaluated on $\{V_i\}_{i \in I}$, and one just has to check that it is compatible with all transition maps (this follows from naturality of the Koszul complex). Taking the quotient of $\text{Sym}(T)$ by the submodule of equations vanishing on the Segre embeddings, we get a Δ -module that sends $\{V_i\}_{i \in I}$ to the homogeneous coordinate ring of the Segre embedding of $\prod_{i \in I} \mathbf{P}(V_i^*)$, call it **Seg**, and its degree d piece **Seg_d**.

The r th homology of the tensor product of the Koszul complex with **Seg** (this computes Tor with the residue field \mathbf{k}) gives the space of r -syzygies of the Segre embedding. If we restrict to a single degree, then each of the terms involved is finitely generated, so the noetherianity result above implies:

Proposition 8.2.4. *For each r, d , the Δ -module given by*

$$\{V_i\}_{i \in I} \mapsto \text{Tor}_r^{\text{Sym}(\bigotimes_{i \in I} V_i)}(\mathbf{Seg}(\bigotimes_{i \in I} V_i), \mathbf{k})_d$$

is finitely generated. Here the subscript d denotes the space of r -syzygies of degree d .

For $r = 1$, this is computing the space of minimal degree d equations of the Segre embedding, so this space is only nonzero for $d = 2$ by our previous discussions. In general, one

can show that the space of r -syzygies is 0 in degrees $> 2r$,⁵ so it is actually superfluous to include the degree d in the above result.

Our original heuristic is that the equations of the Segre embedding are all generated, in some sense, by a single one (the 2×2 determinant). This result says that a similar thing is true if one considers the higher-order syzygies that these equations satisfy: they all come from finitely many basic syzygies. In general, one does not know how to write them all down.

APPENDIX A. REVIEW OF HOMOLOGICAL ALGEBRA

For a thorough treatment of homological algebra, see Weibel’s text [Wei].

A.1. Exact sequences and exact functors. Let \mathcal{A} be an abelian category (the category of R -modules if you like). Suppose we are given a chain complex of objects in \mathcal{A}

$$\mathbf{V} : \cdots \rightarrow V_{i+1} \xrightarrow{f_{i+1}} V_i \xrightarrow{f_i} V_{i-1} \rightarrow \cdots .$$

The i th homology of \mathbf{V} , denoted $H_i(\mathbf{V})$, is

$$H_i(\mathbf{V}) = \ker(f_i) / \text{image}(f_{i+1}).$$

The complex \mathbf{V} is **exact** if $H_i(\mathbf{V}) = 0$ for all i . If our complex has finitely many terms, then we only require exactness everywhere except the end points. In general, elements of $\ker(f_i)$ for some i are **cycles** and elements of $\text{image}(f_i)$ for some i are **boundaries**.

Sequences of the form

$$0 \rightarrow A \xrightarrow{f} B \xrightarrow{g} C \rightarrow 0$$

are called **short exact sequences**. If \mathcal{B} is another abelian category and $F: \mathcal{A} \rightarrow \mathcal{B}$ is a functor (with $F(0) = 0$), we get a corresponding sequence $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$. Then F is

- **exact** if $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$ is an exact sequence for all exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$,
- **left exact** if $0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C)$ is an exact sequence for all exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$,
- **right exact** if $F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow 0$ is an exact sequence for all exact sequences $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$.

Let \mathbf{k} be a commutative ring and let R be a \mathbf{k} -algebra. Let \mathcal{A} be the category of left R -modules and let \mathcal{B} be the category of \mathbf{k} -modules. Given a left R -module M , we have a functor $\text{Hom}_R(M, -)$ given by $N \mapsto \text{Hom}_R(M, N)$. Given a right R -module M , we also have a functor $M \otimes_R -$ given by $N \mapsto M \otimes_R N$.

Proposition A.1.1. • *The functor $\text{Hom}_R(M, -)$ is left-exact.*

- *The functor $M \otimes_R -$ is right-exact.*

Finally, we state the “four lemmas” (for simplicity, just when we’re dealing with the category of R -modules).

⁵This uses the existence of the Taylor complex of a monomial ideal, a semicontinuity argument with Gröbner bases, and the fact that the degree 2 equations for the Segre embedding form a Gröbner basis with respect to some term ordering of the variables.

Lemma A.1.2 (Four lemmas). *Consider the following diagram of R -modules:*

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \alpha \downarrow & & \beta \downarrow & & \gamma \downarrow & & \delta \downarrow & & \varepsilon \downarrow \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

Assume that rows are both exact.

- (a) If β and δ surjective and ε is injective, then γ is surjective.
- (b) If β and δ injective and α is surjective, then γ is injective.

A.2. Derived functors. Let $F: \mathcal{A} \rightarrow \mathcal{B}$ be a left-exact functor. When \mathcal{A} has “enough injectives” (satisfied if \mathcal{A} is the category of left R -modules for some ring R) then there is a sequence of functors $R^i F$ for $i \geq 0$, called the **right derived functors** of F such that

- $R^i F(M) = 0$ whenever M is an injective object,
- $R^0 F = F$,
- For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} , we get a long exact sequence

$$0 \rightarrow F(A) \rightarrow F(B) \rightarrow F(C) \rightarrow R^1 F(A) \rightarrow R^1 F(B) \rightarrow R^1 F(C) \rightarrow R^2 F(A) \rightarrow \cdots .$$

When $F = \text{Hom}_R(M, -)$, we use the notation $R^i F = \text{Ext}_R^i(M, -)$ and $R^i F(N) = \text{Ext}_R^i(M, N)$. These functors commute with taking finite direct sums, i.e., $\text{Ext}_R^i(M, N \oplus N') = \text{Ext}_R^i(M, N) \oplus \text{Ext}_R^i(M, N')$.

There is also a dual situation. Let $G: \mathcal{A} \rightarrow \mathcal{B}$ be a right-exact functor. When \mathcal{A} has “enough projectives” (satisfied if \mathcal{A} is the category of left R -modules for some ring R) then there is a sequence of functors $L_i G$ for $i \geq 0$, called the **left derived functors** of G such that

- $L_i G(M) = 0$ whenever M is a projective object,
- $L_0 G = G$,
- For every short exact sequence $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ in \mathcal{A} , we get a long exact sequence

$$\cdots \rightarrow L_2 G(C) \rightarrow L_1 G(A) \rightarrow L_1 G(B) \rightarrow L_1 G(C) \rightarrow G(A) \rightarrow G(B) \rightarrow G(C) \rightarrow 0.$$

When $G = M \otimes_R -$, we use the notation $L_i G = \text{Tor}_i^R(M, -)$ and $L_i G(N) = \text{Tor}_i^R(M, N)$. These functors commute with arbitrary direct sums, i.e., $\text{Tor}_i^R(M, \bigoplus_\alpha N_\alpha) = \bigoplus_\alpha \text{Tor}_i^R(M, N_\alpha)$.

A.3. (Co)homology of groups. Let Γ be a group. As discussed before, representations of Γ over \mathbf{k} are the same as left $\mathbf{k}[\Gamma]$ -modules. We have the trivial module \mathbf{k} with all elements of Γ acting as the identity, which is both a left and right $\mathbf{k}[\Gamma]$ -module. Given a representation N , we define group homology and cohomology with coefficients in N by

$$\begin{aligned} H_i(\Gamma; N) &= \text{Tor}_i^{\mathbf{k}[\Gamma]}(\mathbf{k}, N), \\ H^i(\Gamma; N) &= \text{Ext}_{\mathbf{k}[\Gamma]}^i(\mathbf{k}, N). \end{aligned}$$

Consider the special case $i = 0$. Then

$$H_0(\Gamma; N) = \mathbf{k} \otimes_{\mathbf{k}[\Gamma]} N \cong N_\Gamma = N / \{n - \gamma n \mid n \in N, \gamma \in \Gamma\}$$

is the coinvariants of N with respect to Γ , and

$$H^0(\Gamma; N) = \text{Hom}_{\mathbf{k}[\Gamma]}(\mathbf{k}, N) = N^\Gamma = \{n \in N \mid \gamma n = n \text{ for all } \gamma \in \Gamma\}$$

is the invariants. If $N = \mathbf{Z}$ with the trivial action, then we usually omit it from the notation. By what we said above, group homology commutes with taking arbitrary direct sums while group cohomology commutes with taking finite direct sums.

If $\mathbf{k}[\Gamma]$ is semisimple, then N is both injective and projective for any representation N , so $H_i(\Gamma; N) = 0$ and $H^i(\Gamma; N) = 0$ for $i > 0$. So this is interesting in the non-semisimple case, e.g., Γ is finite and \mathbf{k} is a field of positive characteristic p where p divides the order of Γ .

As mentioned, $H^i(\Gamma; -)$ and $H_i(\Gamma; -)$ are functors. Suppose we have a group homomorphism $f: \Gamma' \rightarrow \Gamma$. If M is a Γ -module, then it is also a Γ' -module by the action $\gamma' m = f(\gamma') m$ and we have induced maps

$$H_i(\Gamma'; M) \rightarrow H_i(\Gamma; M), \quad H^i(\Gamma; M) \rightarrow H^i(\Gamma'; M),$$

which are compatible with composition of group homomorphisms (but note that cohomology swaps the order of the groups). Given a Γ -equivariant map $\varphi: M \rightarrow N$, we also have a map $H_i(\Gamma; M) \rightarrow H_i(\Gamma; N)$, so we can compose with the above to get

$$H_i(\Gamma'; M) \rightarrow H_i(\Gamma; N).$$

In particular, this comes from a pair of maps $f: \Gamma' \rightarrow \Gamma$ and $\varphi: M \rightarrow N$ such that $\varphi(\gamma' m) = f(\gamma') \varphi(m)$ for all $m \in M$ and $\gamma' \in \Gamma'$. We can do something similar with cohomology.

In the case when the action of $\Gamma = \Gamma'$ and f is conjugation by an element, and $M = \mathbf{k}$ is trivial, the induced map is the identity.

Let $\Omega \subset \Gamma$ be a subgroup. Given an Ω -module M , we have the induction $\text{Ind}_\Omega^\Gamma M = \mathbf{k}[\Gamma] \otimes_{\mathbf{k}[\Omega]} M$ as well as the coinduction $\text{Coind}_\Omega^\Gamma M = \text{Hom}_{\mathbf{k}[\Omega]}(\mathbf{k}[\Gamma], M)$. If Ω is a finite-index subgroup in Γ , then these two are isomorphic.

Lemma A.3.1 (Shapiro's lemma). *With the notation above, we have*

$$H_i(\Gamma; \text{Ind}_\Omega^\Gamma M) = H_i(\Omega; M), \quad H^i(\Gamma; \text{Coind}_\Omega^\Gamma M) = H^i(\Omega; M).$$

A.4. Spectral sequences. While spectral sequences are rather involved structures, we will only emphasize enough to know how they can be applied to our examples of interest. Let \mathcal{A} be an abelian category.

A **homology spectral sequence** (starting with E^a) is the following data:

- An object $E_{p,q}^r$ of \mathcal{A} for every integer $r \geq a$ and pair of integers p, q .
- Morphisms $d_{p,q}^r: E_{p,q}^r \rightarrow E_{p-r, q+r-1}^r$ which are differentials in the sense that $(d^r)^2 = 0$.
- Isomorphisms $E_{p,q}^{r+1} \cong \ker(d_{p,q}^r) / \text{image}(d_{p+r, q-r+1}^r)$.

The degree of an object $E_{p,q}^r$ is the quantity $p + q$, note that d^r always lowers degree by 1. The spectral sequence is **bounded** if for each n , there are only finitely many terms of degree n in E^a .

The sequence of objects with fixed superscript r is the “ r th page” of the spectral sequence, and the last item says that taking homology with respect to the differentials d^r gives the next page. The process of taking homology is called “turning the page”. In our cases of interest, the spectral sequences will be “first quadrant”, meaning that $E_{p,q}^r = 0$ if $p < 0$ or $q < 0$.

For given p, q , if we have $E_{p,q}^r = E_{p,q}^{r+1}$ for $r \gg 0$ (which happens in the bounded case), then we write $E_{p,q}^\infty$ for this limiting value. A bounded spectral sequence **converges** to a collection $\{H_n\}_n$ of objects of \mathcal{A} if each H_n has a finite filtration

$$0 = F_s H_n \subseteq F_{s+1} H_n \subseteq \cdots \subseteq F_{t-1} H_n \subseteq F_t H_n = H_n$$

such that

$$E_{p,q}^\infty \cong F_p H_{p+q} / F_{p-1} H_{p+q}.$$

The notation to indicate a convergent spectral sequence is

$$E_{p,q}^a \implies H_{p+q}.$$

A **cohomology spectral sequence** is similar except we swap superscripts and subscripts and the indices increase instead of decrease and vice versa.

Morally, the purpose of a spectral sequence is to compute the desired quantity H_n from the initial data $E_{p,q}^a$. In practice this is often difficult to carry out, but can be used to gain approximate information. For example, if \mathcal{A} is the category of vector spaces, then in a convergent spectral sequence, we always have an inequality

$$\sum_{p+q=n} \dim_{\mathbf{k}} E_{p,q}^a \geq \dim_{\mathbf{k}} H_n.$$

However, there are other interesting ways to use the existence of a spectral sequence to deduce information. For example, if $H_n = 0$, then it forces maps between various $E_{p,q}^r$ to be isomorphisms.

A general and powerful spectral sequence is the **Grothendieck spectral sequence**. Suppose we have abelian categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and right-exact functors $F: \mathcal{A} \rightarrow \mathcal{B}$ and $G: \mathcal{B} \rightarrow \mathcal{C}$. Then $G \circ F$ is also a right-exact functor. Suppose that \mathcal{A}, \mathcal{B} have enough projectives (automatic if they are categories of modules) and suppose that $(L_i G)(F(M)) = 0$ whenever M is projective. Then we have a spectral sequence for any object M :

$$E_{p,q}^2 = (L_p G)(L_q F)(M) \implies (L_{p+q}(GF))(M).$$

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