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Alfred Bray Kempe's "Proof" of the Four-Color Theorem

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Twenty-five years have passed since Wolfgang Haken and Kenneth Appel provided the mathematics community with a proof of the well-known theorem that any map on a plane or surface of a sphere can be colored with at most four colors so that no two adjacent countries have the same color. Their conquest of the four-color theorem came almost a century after the world had accepted the first "proof" of the theorem. In 1879, Alfred B. Kempe published what he and the mathematics community thought was a proof of the four-color theorem. Unfortunately for Kempe, eleven years later P. J. Heawood discovered a flaw. This article will take a close look at Kempe's attempt to prove the four-color theorem. In addition, we will discuss the conjecture's origin and consider Heawood's counterexample that exposed the flaw in Kempe's work.

Origin of the Conjecture

It has been conjectured that early mapmakers were the first to notice that four colors would suffice when coloring a map. This claim, though logical and tempting to make, has little evidence to support it. In the early 1960's, Kenneth May reviewed a sample of atlases in the large collection of the Library of Congress and found no tendency by mapmakers to minimize the number of colors. May found very few maps that used only four colors, and those that did usually required only three colors. May concluded, "if cartographers are aware of the four-color conjecture, they have certainly kept the secret well."

So when did the four-color conjecture actually arise? Some believe that Francis Guthrie, a British mathematician, was the first person to make the conjecture. In fact, May believed the conjecture "flashed across the mind of Guthrie while he was coloring a map of England." We do know that in 1852 Francis had a conversation with his brother Frederick in which he stated and attempted a proof of the conjecture. Frederick mentions that Francis "showed me the fact that the greatest necessary number of colours to be used in colouring a map so as to avoid identity of colour in lineally contiguous districts is four." Frederick went on to mention that the proof Francis gave "did not seem altogether satisfactory to himself," which probably explains why Francis never published it. Soon after this conversation, Frederick shared the conjecture with Augustus De Morgan, his mathematics professor at University College London.

Sharing the conjecture with De Morgan was a stroke of luck for the mathematics community. De Morgan immediately began to make inquiries about the problem. In an letter to William R. Hamilton, dated October 23, 1852, we have the first written reference to the conjecture. In that letter, De Morgan asked whether Hamilton had heard of the conjecture. Hamilton promptly replied that he had not, and that he would not likely attempt the problem. We know of two other letters from De Morgan in which he discussed the four-color conjecture. In the first letter, dated December 9, 1853, De Morgan wrote to his former teacher, William Whewell; and in the second, dated June 24, 1854, he wrote to Robert Leslie Ellis.

In addition to spreading news of the conjecture through letters and conversations, De Morgan was also responsible for writing the first article that referred to the conjecture. In a book review of Whewell's *The Philosophy of Discovery*, published in the April 14, 1860 issue of the *Athenaeum*, De Morgan included a paragraph that described the four-color conjecture. The paragraph also contained the comment, "it must have been always known to map-colorers that four different colors are enough" which, quite possibly, initiated the tradition of linking the four-color conjecture with cartographers.

Between 1860 and 1878, interest in the four-color problem appeared to wane, and according to Rudolf and Gerda Fritsch, "the problem was not discussed anywhere in the mathematical literature of the time." However, at a meeting of the London Mathematical Society on June 13, 1878, Arthur Cayley appeared to revive interest in the problem when he inquired whether anyone had proved the conjecture. He then published a short paper in which he gave a precise statement of the conjecture and explained where the difficulty was in proving it.

It was not long after Cayley's inquiry that Alfred Bray Kempe, a London barrister and former student of Cayley's, arrived at his now famous and fallacious proof of the four-color conjecture. News of his "proof" was announced in the July 17, 1879 issue of *Nature*, while the full text appeared shortly thereafter in the recently founded *American Journal of Mathematics*.

The "Proof"

There are two observations that should be made when one reads Kempe's paper, observations that may explain why the subtle error in his argument went undetected for eleven years. First, all of his diagrams (there are 16) are relatively simple, and most of them are used to provide examples of the terms he defines. He never provides a nontrivial diagram (map) that demonstrates his argument. Second, the paper is virtually all prose which, though well written, makes it difficult to verify his work.

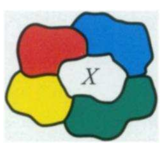
Though the phrase "mathematical induction" was never mentioned in Kempe's paper, the "patching process" he used made his argument essentially a proof by mathematical induction. Therefore, in presenting Kempe's argument, we will use his vocabulary and basic ideas, but we'll give a more contemporary version of his proof.

As with most induction proofs, the base step is quite obvious: any map containing four or fewer countries can easily be colored with at most four colors. Now, assume that any map containing n countries can be colored with at most four colors, and then let M be a map consisting of $n + 1$ countries. It can then be shown—and Kempe did so—that M must contain at least one country that is adjacent to five or fewer other countries. Let X denote such a country in M; then temporarily disregard X. We are left with a map of n countries, which we'll denote by M-X. Now, color the countries of M-X with at most four colors. Let's use red, blue, green, and yellow as Kempe did.

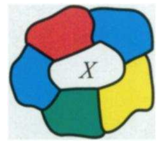
Kempe actually said "take a piece of paper and cut it out to the same shape" as the country X, and then "fasten this patch to the surface and produce all the boundaries which meet the patch to meet at a point within the patch." In other words, Kempe described a process that physically removed the country X and extended the boundaries of the surrounding countries to meet at a point within the region once covered by X.

In the map M-X, we have colored n countries with at most four colors, and we've left X uncolored. Kempe's goal was to find a way to reduce (if necessary) the number of colors used to color the countries surrounding X so that some color would be "free" for X. He quickly dispensed with the easy cases. First, if X is surrounded by three or fewer countries, then clearly there will be a color available for X. Second, if X is surrounded by four or five countries colored with at most three colors, then there will also be a color available for X. With these cases out of the way, Kempe was left with two cases to consider:

Case 1: X is adjacent to exactly four countries colored with four different colors.



Case 2: X is adjacent to exactly five countries colored with four different colors.



In handling these two cases, Kempe used a technique that today we call "the method of Kempe chains." He first asked that we consider all the countries (he called them districts) in the map which are colored red and green; then he observed that these countries form one or more red-green regions. Kempe's notion of a red-green region was simply a continuous "chain" of countries colored red or green. He then made the important observation that one could interchange the colors in any red-green region, and the map would still remain properly colored. We will now demonstrate, using nontrivial examples, the arguments Kempe gave for the two cases.

For case 1, we first label the countries surrounding X with the letters A, B, C, and D. Kempe then considered two subcases.

Subcase 1.1: Suppose countries A and C belong to different red-green regions.

In Figure 1 we have an example of a map in which countries A and C belong to different red-green regions. In this situation, Kempe observed that "we can interchange the colours of the districts in one of these regions, and the result will be that the districts A and C will be of the same colour, both red or both green." By interchanging the colors in the region containing A, we see in Figure 2 that both A and C are now green, making the color red available for X.

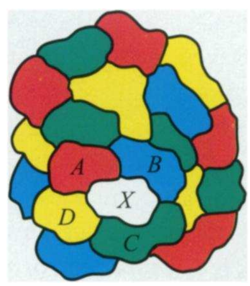


Figure 1

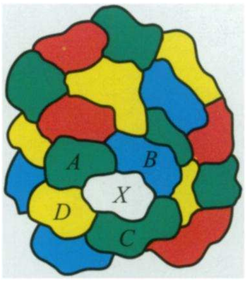


Figure 2

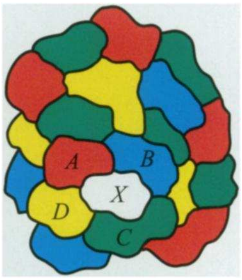


Figure 3

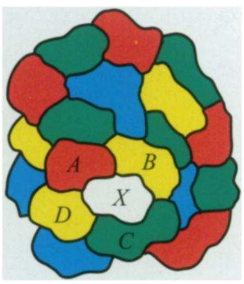


Figure 4

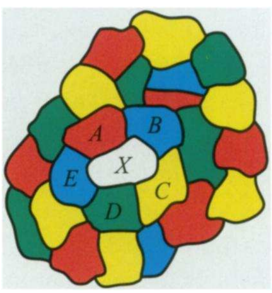


Figure 5

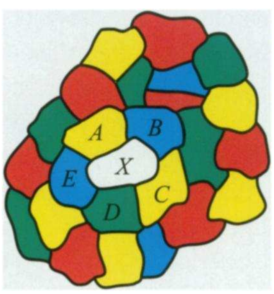


Figure 6

Subcase 1.2: Suppose countries A and C belong to the same red-green region.

In Figure 3 we have an example of a map in which countries A and C belong to the same red-green region. Kempe observed in this case that the red-green region will "form a ring" preventing B and D from belonging to the same blue-yellow region. Therefore, by interchanging the colors in exactly one of these blue-yellow regions, we reduce to three the number of colors surrounding X. In Figure 4 we have interchanged the colors in the blue-yellow region containing B, making the color blue available for X.

For case 2, we label the five countries surrounding X with the letters A, B, C, D, and E. Kempe then considered two subcases.

Subcase 2.1: Suppose we have either countries A and C belonging to different red-yellow regions or countries A and D belonging to different red-green regions.

When one of these alternatives is present in a map, we simply perform an interchange of colors similar to the process used in subcase 1.1. In Figure 5 we have an example of a map in which countries A and C belong to different red-yellow regions. Then, as Kempe claimed, "interchanging the colours in either, A and C become both yellow or both red." If we interchange the red and yellow colors in the region containing A, we obtain the coloring in Figure 6, making red available for X.

Subcase 2.2: Suppose countries A and C belong to the same red-yellow region and countries A and D belong to the same red-green region.

In this, the fourth and final case, Kempe's process for reducing the number of colors surrounding X contained a subtle flaw. In Figure 7 we have an example of a map where countries A and C belong to the same red-yellow region and where countries A and D belong to the

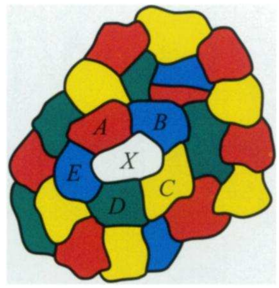


Figure 7

same red-green region. In a case such as this, Kempe correctly observed that "the two regions cut off B from E, so that the blue-green region to which B belongs is different from that to which D and E belong, and the blue-yellow region to which E belongs is different from that to which B and C belong." To reduce the number of colors surrounding X, Kempe then made the claim, "interchanging the colours in the blue-green region to which B belongs, and in the blue-yellow region to which D belongs, B becomes green and E yellow, A, C, and D remaining unchanged." In Figure 8, the interchanges of colors have been performed as Kempe described with the outcome he expected, making the color blue available for X.

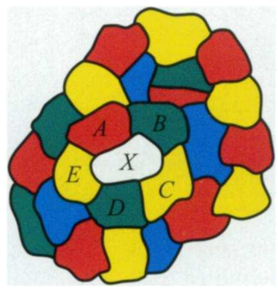


Figure 8

Heawood's Counterexample

In the example used in subcase 2.2, Kempe's process worked exactly as he had hoped. By simultaneously interchanging the colors in the blue-green region containing B and the blue-yellow region containing E, the number of colors surrounding X was reduced to three. Unfortunately for Kempe, this process

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would not work for all maps satisfying the conditions of subcase 2.2.

In 1890, Percy J. Heawood produced a map for which Kempe's process would fail. Heawood's example revealed a subtlety that had escaped detection by the rest of the mathematics community. And that subtlety was the possibility that the blue-green region containing *B* and the blue-yellow region containing *E* might "touch." When this happens, Heawood observed, "Either transposition prevents the other from being of any avail."

In Figure 9 we see the map Heawood used to expose the flaw in Kempe's process for reducing the number of colors in subcase 2.2. Notice that the blue-green region containing *B* and the blue-yellow region containing *E* share a boundary. If

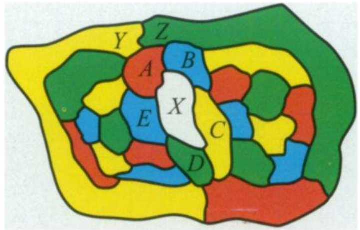

Figure 9

we interchange the colors in both regions, the two countries sharing this boundary, *Y* and *Z*, would both receive the color blue. Thus, as Heawood remarked, "Mr. Kempe's proof does not hold unless some modifications can be introduced into it to meet this case of failure."

Kempe certainly tried to fix this "case of failure," but neither he nor any of his contemporaries could do so. The modifications that were needed would require many years of work by many individuals.

Conclusion

The importance of Kempe's work cannot be overlooked. His basic ideas provided the starting point for what would be a century of effort culminating with Appel and Haken's proof. In 1989, as a tribute to Kempe, Appel and Haken declared: "Kempe's argument was extremely clever, and although his "proof" turned out not to be complete, it contained most of the basic ideas that eventually led to the correct proof one century later." ■

For Further Reading

Interested readers will find a detailed history of the four-color problem and a thorough list of the relevant literature in *The Four-Color Theorem: History, Topological Foundations, and Idea of Proof* by Rudolf and Gerda Fritsch.