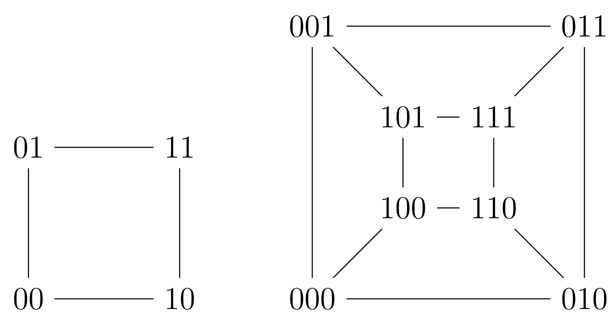
Math 154, Winter 2019
Homework 4
Due: Monday, February 11 by 5PM in basement of AP&M

(1) If G is a simple graph with n vertices, define its **degree sequence** to be the list of the degrees $(d_1, \dots, d_n)$ of its vertices in weakly increasing order (so $d_1 \le d_2 \le \cdots \le d_n$).
 (a) Prove that $(d_1 + d_2 + \cdots + d_n)/2$ is the number of edges of G.
 (b) Prove that if G and H are isomorphic simple graphs, then their degree sequences are the same.
 (c) Show that the converse need not be true by finding two simple graphs on 6 vertices with degree sequence $(2, 2, 2, 2, 2, 2)$ which are not isomorphic to each other.

(2) Draw all isomorphism classes of simple graphs with 4 vertices. You may use the fact that there are 11: so just produce 11 graphs, but you should explain why your list doesn't have any repetitions.

(3) Let n be a positive integer. Define a simple graph Q_n as follows:
- The vertices are n-tuples $(x_1, \dots, x_n)$ with $x_i \in \{0, 1\}$.
- There is an edge between $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ if they agree in exactly $n - 1$ coordinates (i.e., there exists i such that $x_j = y_j$ if $j \ne i$ but $x_i \ne y_i$).

Here are drawings of Q_2 and Q_3:



 (a) Prove that Q_n is connected.
 (b) How many vertices does Q_n have? How many edges?
 (c) For what values of n does Q_n have a closed Eulerian trail?
 (d) Prove that if $n \ge 2$, then Q_n has a Hamiltonian cycle.
 (e) Let σ be a permutation of $[n]$. Show that the function $(x_1, \dots, x_n) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ is an automorphism of Q_n. Give an example (for all n) of another automorphism of Q_n which is not of this form.

(4) Let G be a graph and assume there is a walk from vertex A to vertex B. Prove that there is also a path from A to B. (Remember, a path is a walk that doesn't touch any vertex more than once.)

Hints:
2: Organize them by number of edges; #1(b) might be helpful
4: Among all walks from A to B, consider one whose length is as small as possible.