Math 184, Fall 2019
Homework 3
Due: Friday, Nov. 1 by 3:00PM in homework box #2 in basement of AP&M
(late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences. Some hints are on the last page.

(1) Evaluate the following sums:

(a) $\displaystyle\sum_{i=0}^{n} \binom{n}{i} \frac{1}{2^i}$

(b) $\displaystyle\sum_{i=0}^{n} i \binom{n}{i} 3^i$

(2) Fix positive integers n, m, k. Prove that

$$\sum_{i=0}^{k} \binom{n}{i} \binom{m}{k-i} = \binom{n+m}{k}.$$

(3) Let $n \geq 2$ be an integer.

(a) Prove that

$$\sum_{i=0}^{n} i \binom{n}{i} (-1)^{i-1} = 0.$$

(b) Deduce from (a) that

$$\sum_{\substack{0 \leq i \leq n \\ i \text{ odd}}} i \binom{n}{i} = \sum_{\substack{0 \leq i \leq n \\ i \text{ even}}} i \binom{n}{i}$$

and compute the common value.

(4) (a) Using the multinomial theorem, compare the coefficients of both sides of the equation $(x+y+z)(x+y+z)^n = (x+y+z)^{n+1}$ to get a generalization of Pascal's identity for multinomial coefficients.

(b) Do the same thing with k variables for general k.

(5) A "forward path" in the plane is a sequence of steps of the form $(1, 0)$ and $(0, 1)$.

(a) How many forward paths are there from $(0, 0)$ to (a, b) where a, b are non-negative integers?

(b) Let $S_{a,b}$ be the set of integer partitions λ such that $\ell(\lambda) \leq b$ and $\lambda_1 \leq a$. Find a bijection between $S_{a,b}$ and the set of forward paths from $(0, 0)$ to (a, b).

(c) Generalize this definition to d dimensions by only allowing steps which increase one of the coordinates by 1 (so $(1, 0, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, 0, \ldots, 1)$). How many forward paths are there from $(0, 0, \ldots, 0)$ to $(a_1, a_2, \ldots, a_d)$ where $a_1, \ldots, a_d$ are non-negative integers?

Hints:

5b: Draw a rectangle with endpoints $(0,0)$, $(a,0)$, (a,b), $(0,b)$. Think of a forward path as splitting this rectangle into two pieces and consider the portion above the path.