Math 184, Fall 2019
Homework 1
Due: Friday, Oct. 11 by 3:00PM in homework box #2 in basement of AP&M
(late homework will not be accepted)

Explanations should be given for your solutions. Use complete sentences.

(1) Prove that every polynomial in x can be written as a linear combination of the polynomials

$$1, 2x-1, (2x-1)^2, (2x-1)^3, (2x-1)^4, \ldots.$$

(2) How many ways are there to list the letters of the word MATHEMATICIAN?

(3) How many integers are there between 10000 and 99999 in which all digits are different?

(4) Let $n \geq 3$ be an integer. Define the following sets:

$$\begin{aligned} A &= \{S \subseteq [n] \mid 1 \in S \text{ and } 3 \in S\}, \\ B &= \{S \subseteq [n] \mid 1 \in S \text{ and } 3 \notin S\}, \\ C &= \{S \subseteq [n] \mid 1 \notin S \text{ and } 3 \notin S\}, \\ D &= \{S \subseteq [n] \mid |\{1,3\} \cap S| \geq 1\}. \end{aligned}$$

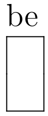
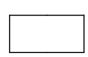
Find formulas for the size of each set.

(5) (a) We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$ and $B \subseteq C$. How many ways can this be done?
(b) We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$. How many ways can this be done?

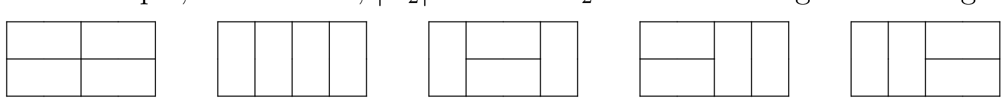
(6) Fix a positive integer $n \geq 1$. Let A_1 be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i+1 \notin S$.

For example, when $n = 3$, $|A_1| = 5$ and A_1 is the following set of subsets:

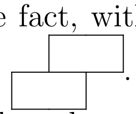
$$\emptyset, \{1\}, \{2\}, \{3\}, \{1,3\}.$$

Let A_2 be the set of ways of tiling the $2 \times (n+1)$ rectangle with the shapes: 2×1 rectangle  and 1×2 rectangle  without any overlaps.

For example, when $n = 3$, $|A_2| = 5$ and A_2 is the following set of tilings:



Construct a bijection between A_1 and A_2 (and prove that it is a bijection).

You may use the fact, without proving it, that the following configuration never appears in a tiling: .

Hint: Consider the column indices where there are horizontal tiles.