Math 184A, Fall 2018
Homework 4
Due: Friday, November 2 by 3:30PM in basement of AP&M

There are some hints on the next page.

(1) How many positive integers ≤ 1000 are neither perfect squares nor perfect cubes? [Recall that a perfect square is an integer of the form n^2 where n is an integer, and a perfect cube is an integer of the form n^3 where n is an integer.]

(2) Let λ be an integer partition. Write $\lambda \subseteq m \times n$ if $\ell(\lambda) \leq m$ and $\lambda_1 \leq n$, i.e., the Young diagram of λ fits inside of a $m \times n$ rectangle. For $0 < k < n$, define a polynomial $P_{n,k}(x)$ by

$$P_{n,k}(x) = \sum_{\lambda \subseteq k \times (n-k)} x^{|\lambda|}.$$

In other words, the coefficient of x^i is the number of partitions of i whose Young diagram fits into the $k \times (n-k)$ rectangle. By convention, $P_{n,n}(x) = P_{n,0}(x) = 1$. As an example, $P_{4,2}(x) = 1 + x + 2x^2 + x^3 + x^4$ (the 1 corresponds to the fact that there is a single partition of size 0).

(a) Show that $P_{n,k}(x) = P_{n,n-k}(x)$.

(b) If $0 < k < n$, show that

$$P_{n,k}(x) = x^k P_{n-1,k}(x) + P_{n-1,k-1}(x).$$

(c) Using (b), show that $P_{n,k}(1) = \binom{n}{k}$ for all $0 \leq k \leq n$.
[If you cannot solve (b), you can still use it to solve this problem for credit.]

(d) Find a direct explanation (not using (b)) for why $P_{n,k}(1) = \binom{n}{k}$. In other words, show that the number of Young diagrams that fit inside the $k \times (n-k)$ rectangle is $\binom{n}{k}$.

(3) How many ways are there to list the letters of the word WISCONSIN so that no two consecutive letters are the same?

(4) Fix a positive integer n. Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

(5) Fix a positive integer n. Show that

$$\sum_{k=0}^{n} \frac{(-1)^{k+1}}{k+1} \binom{n}{k} = -\frac{1}{n+1}.$$

Hints:

2(b): Think about adding/removing columns from Young diagrams

2(c): You can either do a double induction, or do induction on $n+k$.

2(d): Given a Young diagram $Y(\lambda) \subseteq k \times (n-k)$, we can remove it, and the top boundary of the resulting shape is a path from the bottom left corner of the rectangle to the top right corner using the steps "up" and "right". Show these are counted by $\binom{n}{k}$.

5: Antiderivative