Math 184A, Fall 2018
Homework 2
Due: Friday, Oct. 12 by 3:30PM in basement of AP&M

(1) How many solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 = 96$$

where each x_i is required to be a nonnegative integer?

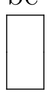
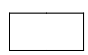
(2) How many ways are there to list the letters of the word LAJOLLA?

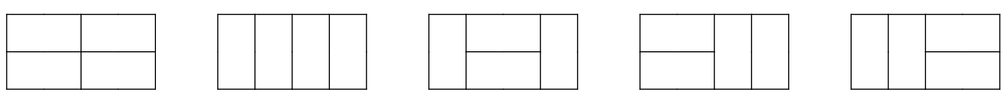
(3) How many integers are there between 1000 and 9999 in which all digits are different?

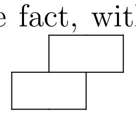
(4) (a) We want to select two subsets A and C of $[n]$ so that $A \subseteq C$. Show that there are 3^n ways to do this.
(b) We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$ and $B \subseteq C$. How many ways can this be done?
(c) We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$. How many ways can this be done?

(5) Fix a positive integer $n \geq 1$. Let A_1 be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i + 1 \notin S$.
For example, when $n = 3$, $|A_1| = 5$ and A_1 is the following set of subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}.$$

Let A_2 be the set of ways of tiling the $2 \times (n + 1)$ rectangle with the shapes: 2×1 rectangle  and 1×2 rectangle  without any overlaps.
For example, when $n = 3$, $|A_2| = 5$ and A_2 is the following set of tilings:



Construct a bijection between A_1 and A_2 (and prove that it is a bijection).
You may use the fact, without proving it, that the following configuration never appears in a tiling: 

(6) Let n and k be positive integers. Show that the number of ordered collections $(X_1, \ldots, X_k)$, where each X_i is a subset of $[n]$, and $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$ (i.e., there is no element which is in all of the X_i) is $(2^k - 1)^n$.
For example, when $k = 2$ and $n = 2$, here are the 9 ordered collections:

$$\begin{array}{ccc} (\emptyset, \emptyset) & (\emptyset, \{1\}) & (\emptyset, \{2\}) \\ (\emptyset, \{1, 2\}) & (\{1\}, \emptyset) & (\{2\}, \emptyset) \\ (\{1, 2\}, \emptyset) & (\{1\}, \{2\}) & (\{2\}, \{1\}). \end{array}$$