MATH 740 HOMEWORK
SPRING 2017

Last updated: April 14, 2017

These are due by noon on May 10. The list will grow over the course of the semester and your grade in the course will be determined by how many you correctly solve (an A will be roughly 80%).

I'll put point values on the problems below since some problems are harder/longer than others. The total points so far is 9.

These problems are meant to help you understand the material. Please do not consult any references besides the notes, myself, and other students. In particular, you may turn these in at any time for feedback. You can then resubmit solutions based on that feedback.

(1) (1 point) Give a combinatorial proof (using SSYT) that

$$s_\lambda(x_1, y_1, x_2, y_2, \dots) = \sum_{\mu \subseteq \lambda} s_\mu(x_1, x_2, \dots) s_{\lambda/\mu}(y_1, y_2, \dots).$$

(2) (1 point) Given a partition λ, let 2λ denote the partition obtained by doubling all of its entries. Use the Pieri rule to show that

$$\left(\sum_\lambda s_{2\lambda}\right)\left(\sum_{d \geq 0} e_d\right) = \sum_\mu s_\mu.$$

The first and third sums are both over all partitions.

(3) (3 points) Let λ, μ, ν be partitions of n. The product $\chi^\lambda \chi^\mu$ (naive product, not induction product!) is the character of the tensor product of the representations corresponding to λ and μ, and hence is a character; so we can decompose it:

$$\chi^\lambda \chi^\mu = \sum_\nu g^\nu_{\lambda,\mu} \chi^\nu$$

These coefficients are called **Kronecker coefficients**. These are notoriously difficult to understand, here we will examine a few basic properties.

(a) Show that g is invariant under permutations of ν, λ, μ, i.e.,

$$g^\nu_{\lambda,\mu} = g^\nu_{\mu,\lambda} = g^\lambda_{\nu,\mu} = g^\lambda_{\mu,\nu} = g^\mu_{\lambda,\nu} = g^\mu_{\nu,\lambda}.$$

For that reason, they are usually written more symmetrically as $g_{\lambda,\mu,\nu}$.

(b) Using the Frobenius characteristic ch, we can transfer the product of characters to symmetric functions: given $f, g \in \Lambda$, define

$$f * g = \mathrm{ch}(\mathrm{ch}^{-1}(f)\mathrm{ch}^{-1}(g)).$$

For any character χ, show that $\mathrm{ch}(\chi) * p_\lambda = \chi(\lambda) p_\lambda$.

(c) We showed that $p_1, p_2, \dots$ are algebraically independent and hence every $f \in \Lambda_{\mathbf{Q}}$ can be written as a polynomial in the p_n with $\mathbf{Q}$-coefficients. Define $\frac{\partial}{\partial p_n} f$ to be partial derivative of this polynomial where we are treating the p_n as variables, so for example, $\frac{\partial}{\partial p_2}(p_2^2 p_5 + p_1 + p_1 p_2^3) = 2p_2p_5 + 3p_1p_2^2$. Show that for all $f, g \in \Lambda$, we have

$$\left\langle n\frac{\partial}{\partial p_n} f, g \right\rangle = \langle f, p_n g\rangle.$$

(d) Let Y^1 be the character of the permutation representation of Σ_n on $\mathbf{C}^n$, i.e., $Y^1(\mu)$ is the number of fixed points of μ. Show that $\mathrm{ch}(Y^1) * s_\lambda = s_1 s_{\lambda/1}$.
Hint: first show that $\mathrm{ch}(Y^1) * p_\lambda = m_1(\lambda)p_\lambda = p_1\frac{\partial}{\partial p_1}p_\lambda$ and show that $\frac{\partial}{\partial p_1}s_\lambda = s_{\lambda/1}$ by showing they pair the same way against all symmetric functions.

(e) Use $Y^1 = \chi^{(n-1,1)} + \chi^n$ to deduce a formula for $g_{(n-1,1),\lambda,\mu}$.

(4) (1 point) Here's another appearance of Kronecker coefficients. Consider the identification

$$(\mathbf{C}^m \otimes \mathbf{C}^n)^{\otimes d} = (\mathbf{C}^m)^{\otimes d} \otimes (\mathbf{C}^n)^{\otimes d}.$$

This is an isomorphism of representations of $\mathbf{GL}_m(\mathbf{C}) \times \mathbf{GL}_n(\mathbf{C}) \times \Sigma_d$. Apply Schur–Weyl duality to all 3 tensor powers to conclude that

$$\mathbf{S}_\lambda(\mathbf{C}^m \otimes \mathbf{C}^n) \cong \bigoplus_{\mu,\nu}(\mathbf{S}_\mu(\mathbf{C}^m) \otimes \mathbf{S}_\nu(\mathbf{C}^n))^{\oplus g_{\lambda,\mu,\nu}}.$$

Conclude from this and the Cauchy identity the following identity for symmetric functions in 3 different sets of variables x, y, z:

$$\prod_{i,j,k}(1 - x_i y_j z_k)^{-1} = \sum_{\lambda,\mu,\nu} g_{\lambda,\mu,\nu} s_\lambda(x) s_\mu(y) s_\nu(z)$$

where the product is over all positive integers i, j, k and the sum is over all triples of partitions.

(5) (1 point) Use Schubert calculus to determine the number of lines in a sufficiently generic quintic hypersurface in $\mathbf{P}^4$. In other words, given a generic homogeneous quintic (degree 5) equation f in 5 variables, determine the number of 2-dimensional linear spaces W such that f is the zero function on W.

(6) (2 points) Finish the proof of Proposition 9.3.1 from the notes: show that $\{\rho(\xi_A)\}$ is linearly independent in $M_2(\mathbf{C})^{\otimes k}$.

Changelog:

- January 31: created file
- February 14: added problems 3 and 4
- March 2: added problems 5 and 6
- March 7: minor correction to 3e.
- March 15: removed problem 6
- April 14: added new problem 6