MATH 490 HOMEWORK 2
DUE: FEBRUARY 9

Your homework should be written in LaTeX.

(1) In class, we computed the homology of a connected graph. Generalize this computation to an arbitrary (finite) graph.

(2) Let F be a field and let ϕ be an $m \times n$ matrix with entries in F.

Given a non-negative integer k, a $k \times k$ minor of ϕ is the determinant of a $k \times k$ submatrix of ϕ. Prove that the rank of ϕ is r if and only if all $(r+1) \times (r+1)$ minors of ϕ are 0, and some $r \times r$ minor is nonzero.

(The determinant of a 0×0 matrix is defined to be 1.)

(3) Consider the following chain complex[1]:

$$\mathbf{Z}^2 \xrightarrow{\begin{pmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}} \mathbf{Z}^3 \xrightarrow{\begin{pmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \end{pmatrix}} \mathbf{Z}^2$$

(a) Show that $\mathrm{H}_2 \cong 0$, $\mathrm{H}_1 \cong \mathbf{Z}/2$, and $\mathrm{H}_0 \cong \mathbf{Z}$.

(b) Use problem 2 to compute the ranks of the two matrices with $\mathbf{Z}$ replaced by the field of rational numbers $\mathbf{Q}$ and also with the finite field $\mathbf{Z}/p$ where p is a prime.

(c) Use (b) to compute the dimensions of the homology groups when $\mathbf{Z}$ is replaced by one of the fields $\mathbf{Q}$ or $\mathbf{Z}/p$.

[1]This computes the homology of the real projective plane $\mathbf{RP}^2$ but that fact isn't needed.

Hints:

(1) Different connected components of a graph don't interact with each other when applying the boundary map.
(2) The rank of a matrix is the dimension of its column space, and also the dimension of its row space. A square matrix has determinant 0 if and only if its columns are linearly dependent if and only if its rows are linearly independent.
(3) For (a), to calculate H_1, first find a spanning set for the kernel of the right map and a spanning set for the image of the left map. Then show there are exactly 2 different cosets when quotienting the kernel by the image.