MATH 490 EXERCISES

Here is a list of things to think about based on topics that arise in lecture. You don't need to turn these in. However, to gain familiarity with the subjects, you'll want to think about how to solve these.

1. January 17

(1) Verify that $\mathbf{Z}/n$ is a commutative ring.

(2) Show that $\mathbf{Z}/p$ is a field when p is a prime. More specifically, show that if $a \not\equiv 0 \pmod{p}$, then there exists x such that $ax \equiv 1 \pmod{p}$.

Hint: use the Euclidean algorithm.

(3) Let R be a commutative ring and let M and N be R-modules. Recall that $\mathrm{Hom}_R(M, N)$ is the set of homomorphisms $f\colon M \to N$. For $r \in R$, $f, g \in \mathrm{Hom}_R(M, N)$, define $f+g$ by $(f+g)(m) = f(m)+g(m)$ and rf by $(rf)(m) = rf(m)$. Verify that $\mathrm{Hom}_R(M, N)$ is an R-module.

2. January 19

(1) Let R be a commutative ring. Let $f\colon M \to N$ be a homomorphism of R-modules.

(a) Verify that $\mathrm{im}(f)$ is a submodule of N.

(b) Construct a short exact sequence

$$0 \to \ker f \to M \to \mathrm{im}(f) \to 0.$$

(2) Let R be a field.

(a) Let

$$0 \to M_2 \xrightarrow{d_1} M_1 \xrightarrow{d_0} M_0 \to 0$$

be a short exact sequence of vector spaces. Show that

$$\dim M_0 - \dim M_1 + \dim M_2 = 0.$$

Hint: Show that $M_2 \cong \mathrm{im}(d_1)$ and that $M_0 \cong M_1/\mathrm{im}(d_1)$.

(b) Consider a longer complex

$$0 \to M_3 \xrightarrow{d_2} M_2 \xrightarrow{d_1} M_1 \xrightarrow{d_0} M_0 \to 0$$

which is exact everywhere. Show that

$$\dim M_0 - \dim M_1 + \dim M_2 - \dim M_3 = 0.$$

Hint: Split this up into two short exact sequences by applying 1(b) to d_1 and d_0 and then apply 2(a) to both.

(c) Formulate and prove a general statement for longer exact complexes.

3. January 24

(1) We defined a bijection $\mathrm{Hom}_R(R^n, M) \to M^n$ which sends a homomorphism f to $(f(e_1), f(e_2), \ldots, f(e_n))$. Show that this function is in fact an R-module homomorphism.

(2) We explained how to turn a linear map $f\colon R^n \to R^m$ into an $m \times n$ matrix $M(f)$. Verify that given $g\colon R^m \to R^p$, we have $M(g \circ f) = M(g)M(f)$, i.e., that composition turns into matrix multiplication.