Math 847
Homework 1
Due: November 21, 2017

Work in groups of up to 3. It is preferable if you type your solutions. Minor details do not need to be explained, use your best judgement. The point here is to make sure that you engage with the material and have learned something.

1. Polynomiality of FI-modules. For this exercise, we consider **FI**-modules over a field $\mathbf{k}$. Let M be an **FI**-module. Define a new **FI**-module ΣM by $(\Sigma M)(S) = M(S \amalg \{*\})$, where $S \amalg \{*\}$ means we have taken the disjoint union with a new element. There is a natural map $M \to \Sigma M$ coming from the inclusion of S into $S \amalg \{*\}$. Let ΔM be the cokernel.

(1) Let K be the kernel of $M \to \Sigma M$. Show that $K(f) = 0$ for all non-bijective injections f and deduce that if M is finitely generated, then $K(S) = 0$ for all finite sets with $|S| \gg 0$.
(2) If M is generated in degrees $\le d$, show that ΔM is generated in degrees $\le d-1$. Deduce that $\Delta^{d+1} M = 0$.
(3) Conclude that if M is finitely generated in degrees $\le d$, then the function $n \mapsto \dim_{\mathbf{k}} M([n])$ is a polynomial of degree $\le d$ for $n \gg 0$.

2. VI-modules. Let q be a prime power and $\mathbf{F}_q$ be the finite field with q elements. Define a category $\mathbf{VI}(\mathbf{F}_q)$ whose objects are finite-dimensional vector spaces over $\mathbf{F}_q$ and whose morphisms are linear injections. Show that the category of $\mathbf{VI}(\mathbf{F}_q)$-modules over a field $\mathbf{k}$ is locally noetherian.

Hint. Use the strategy from class for proving that the category of $\mathbf{FI}_d$-modules over a field is locally noetherian by finding a suitable definition of an ordered version of this category $\mathbf{OVI}(\mathbf{F}_q)$ together with a functor $\Phi\colon \mathbf{OVI}(\mathbf{F}_q) \to \mathbf{VI}(\mathbf{F}_q)$ that satisfies property (F).