Comments on Eisenbud, Exercise 17.21
February 26, 2016

Recall the setup: k is a (commutative) ring and $V \cong k^n$ is a finitely generated free k-module. $S(V)$ is the symmetric algebra on V and $\bigwedge(V^*)$ is the exterior algebra on the dual V^*. For simplicity, we will just assume that k is a field.

It is convenient to think of elements of V as having degree 1 and elements of V^* as having degree -1. Then these algebras both have graded decompositions:

$$S(V) = \bigoplus_{d \geq 0} S^d(V),$$

$$\bigwedge(V^*) = \bigoplus_{d \geq 0} \bigwedge^d(V^*),$$

(so $S^d(V)$ has degree d while $\bigwedge^d(V^*)$ has degree $-d$).

The element $t \in \bigwedge^1(V^*) \otimes S^1(V) = V^* \otimes V$ is the canonical trace element, i.e., pick a basis e_i for V and a dual basis e_i^* for V^* and take $t = \sum_{i=1}^n e_i^* \otimes e_i$; this is independent of the choice of basis and t has degree 0.

In particular, multiplication by t is a degree-preserving operator on $\bigwedge(V^*) \otimes S(V)$ and squares to 0 and we can identify the complex

$$0 \to S(V) \xrightarrow{\cdot t} V^* \otimes S(V) \xrightarrow{\cdot t} \bigwedge^2(V^*) \otimes S(V) \xrightarrow{\cdot t} \cdots \xrightarrow{\cdot t} \bigwedge^n(V^*) \otimes S(V)$$

with the Koszul complex on the elements $e_1, \dots, e_n \in S(V)$. We've already seen that this is exact except at the right end (since $e_1, \dots, e_n$ is a regular sequence) at which case the homology is k (concentrated in degree $-n$ since it's a quotient of $\bigwedge^n(V^*) \otimes S(V)$ which is a free $S(V)$-module of rank 1 generated in degree $-n$).

Since t is degree-preserving, we can decompose this into linear strands, i.e., for every d, you get a complex of k-modules of degree d:

$$(1) \qquad 0 \to S^d(V) \to V^* \otimes S^{d+1}(V) \to \bigwedge^2(V^*) \otimes S^{d+2}(V) \to \cdots \to \bigwedge^n(V^*) \otimes S^{d+n}(V)$$

which is also valid for $d < 0$ if we interpret $S^d(V) = 0$ in this case. This complex is always exact and if $d \neq -n$, the last map is also surjective (this is just reformulating what we just said about the Koszul complex).

Multiplication by t also gives us this complex considered in (c):

$$(2) \qquad \bigwedge(V^*) \to \bigwedge(V^*) \otimes S^1(V) \to \bigwedge(V^*) \otimes S^2(V) \to \cdots,$$

and again since it is degree-preserving, for any e we get a complex of k-modules of degree $-e$:

$$(3) \qquad \bigwedge^e(V^*) \to \bigwedge^{e+1}(V^*) \otimes S^1(V) \to \bigwedge^{e+2}(V^*) \otimes S^2(V) \to \cdots.$$

This is the same complex as in (1) if we take $d = -e$ (again, interpret negative exterior powers to be 0).

We know that (1) is exact if $d \neq -n$, so (3) is exact if $e \neq n$. In particular, we can extend (2) to an exact complex as follows:

$$0 \to k \to \bigwedge(V^*) \to \bigwedge(V^*) \otimes S^1(V) \to \bigwedge(V^*) \otimes S^2(V) \to \cdots, \tag{4}$$

where k is concentrated in degree $-n$.

To calculate $\mathrm{Ext}^*_{\bigwedge(V^*)}(k,k)$, use that $\bigwedge(V^*)$ is self-injective (i.e., injective as a module over itself, see Proposition 1 below) and so (4) is an injective resolution of k. Now apply $\mathrm{Hom}_{\bigwedge(V^*)}(k,-)$ to it and notice that all differentials become 0 by degree reasons, so

$$\mathrm{Ext}^i_{\bigwedge(V^*)}(k,k) = \mathrm{Hom}_{\bigwedge(V^*)}(k, \bigwedge(V^*) \otimes S^i(V)) = S^i(V)$$

(here we use that a $\bigwedge(V^*)$-linear map from k to $\bigwedge(V^*)$ must land in $\bigwedge^n(V^*)$).

Alternatively, you can do something with duals if you wanted to use a projective resolution of k as a $\bigwedge(V^*)$-module, but that gets a little bit more messy.

Proposition 1. *$\bigwedge(V^*)$ is self-injective.*

Proof. $\mathrm{Hom}_k(\bigwedge(V^*), k)$ is an injective module over $\bigwedge(V^*)$ (see Lemma A3.8 of Eisenbud, for example). Fix a generator $z \in \bigwedge^n(V^*)$. Define a pairing on $\bigwedge(V^*)$ by setting $\beta(a,b)$ to be the coefficient of z in $a \wedge b$ (in the homogeneous decomposition of $a \wedge b$). Then β is a perfect pairing. Define a map $\bigwedge(V^*) \to \mathrm{Hom}_k(\bigwedge(V^*), k)$ by $a \mapsto f_a$ where $f_a(b) = \beta(b,a)$. Note that $f_{a' \wedge a}(b) = \beta(b, a' \wedge a) = \beta(b \wedge a', a) = f_a(b \wedge a')$ so this is a module homomorphism. Since β is a perfect pairing, it's also an isomorphism. So we conclude that $\bigwedge(V^*)$ is self-injective. □