Math 746, Spring 2016
Homework 3
Due: March 18

## 1. Exercises

(1) Eisenbud 19.10
(2) Eisenbud 19.15

(3) Let $\phi$ be a skew-symmetric matrix (i.e., $\phi_{ii} = 0$ and $\phi_{ij} = -\phi_{ji}$) of odd size $2n+1$ with entries in $R$. Let $f_j$ be the Pfaffian[1] of the skew-symmetric submatrix obtained by deleting the $j$th column and $j$th row of $\phi$. Let $J(\phi)$ be the ideal generated by $f_1, \dots, f_{2n+1}$. Define

$$\mathbf{F}\colon 0 \to R \xrightarrow{\phi_3} R^{2n+1} \xrightarrow{\phi_2} R^{2n+1} \xrightarrow{\phi_1} R$$

as follows:
- $\phi_1 = \begin{pmatrix} f_1 & -f_2 & \cdots & -f_{2n} & f_{2n+1} \end{pmatrix}$,
- $\phi_2 = \phi$,
- $\phi_3 = \phi_1^T$, where $T$ denotes transpose.

(a) Verify that $\mathbf{F}_\bullet$ is a complex.
(b) Consider the universal case $R = \mathbf{Z}[x_{ij} \mid 1 \le i < j \le 2n+1]$ and

$$\phi_{ij} = \begin{cases} 0 & \text{if } i = j \\ x_{ij} & \text{if } i < j \\ -x_{ij} & \text{if } i > j \end{cases}.$$

Show that depth $J(\phi) = 3$ and that $\mathbf{F}_\bullet$ is exact.
(c) Go back to the general case of a noetherian ring $R$ and assume that $J(\phi) \neq R$. Use the generic perfection theorem to show that depth $J(\phi) \le 3$ and that $\mathbf{F}_\bullet$ is exact if and only if depth $J(\phi) = 3$.

(4) Let $k$ be a field and let $I \subset k[x, y, z]$ be the ideal of polynomials vanishing on a finite set of points in $\mathbf{P}^2_k$.[2] Use the Hilbert–Burch theorem to show that if the points lie on a curve of degree $d$, then $I$ can be generated by $d+1$ elements.

## 2. Further reading

The complex in #3 was studied in Buchsbaum, Eisenbud, "Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3" and plays a role similar to the Hilbert–Burch complex. Namely, assuming that the ideal $J(\phi)$ has depth 3, the ring $R/J(\phi)$ is Gorenstein, which we will learn about later in Chapter 21. Conversely, a depth 3 ideal whose quotient is Gorenstein and which has a finite free resolution of length 3 must be of the form $J(\phi)$ for some skew-symmetric matrix $\phi$ (up to a choice of basis, $\phi$ is the second map in the free resolution) – the existence of the resolution is automatic if $R$ is a regular (graded) local ring.

[1] Recall that if $X$ is a skew-symmetric matrix of even size, there is a function, the Pfaffian, such that $\mathrm{Pf}(X)^2 = \det(X)$. This has a Laplace expansion formula similar to the determinant case; the wikipedia page has many basic properties, or see any introductory algebra text.

[2] Algebraically, this means that $I$ is radical and $R/I$ is equidimensional of dimension 1.