Math 742, Spring 2016
Homework 7
Due: March 14 (MONDAY)

1. Exercises

(1) Let k be a field and $R = k[x_1, \dots, x_n]$. Let I be a monomial ideal.
   (a) Let m be a monomial which is a minimal generator of I and suppose we have a factorization into other monomials $m = m'm''$. Show that $I = (I+(m')) \cap (I+(m''))$.
   (b) Use (a) to describe a primary decomposition of a monomial ideal into primary monomial ideals.
   (c) In the case $R = k[x, y, z]$, find a primary decomposition of (x^2, xz^3, yz). Identify the minimal and embedded associated primes.

(2) Let k be a field and let R be a finitely generated k-algebra. Show that R is artinian if and only if R is a finite-dimensional vector space over k.

(3) The results of Chapter 19 show that any artinian commutative ring is a direct product of artinian local rings. In particular, any finite commutative ring is a direct product of finite local rings. We will see that this is false in the non-commutative setting. Let R be a nonzero ring (not necessarily commutative).

   Some terminology: $x \in R$ is **left-invertible** if there exists $y \in R$ such that $yx = 1$. **Right-invertible** is defined in a similar way. An element is a **unit** if it is both left- and right-invertible. A **left ideal** I is a subset closed under addition such that $x \in R$ and $y \in I$ implies $xy \in I$. A left ideal I is **maximal** if it is not the whole ring and any other left ideal that contains I is either the whole ring or equal to I. The notion of **right ideal** and maximal right ideals is similar.
   (a) Assume that $xy = 1$ and $yx \neq 1$. Use the identity $yx(1 - yx) = (1 - yx)yx = 0$ to show that yx and $1 - yx$ are not left-invertible and not right-invertible.
   (b) Show that if R has a unique maximal left ideal, then this ideal is also a right ideal and, in fact, is the unique maximal right ideal. Also, show that this is the set of non-units of R. In this case we say that R is **local**. Conclude that in a local ring, left-invertibility and right-invertibility are equivalent.
   (c) $e \in R$ is a **central idempotent** if $e^2 = e$ and $ex = xe$ for all $x \in R$, and it is nontrivial if $e \neq 0$ and $e \neq 1$. Show that R is isomorphic to a direct product of two nonzero rings if and only if there exists a nontrivial central idempotent $e \in R$.
   (d) Let k be a commutative field. Let
$$R = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \middle| \, a, b, c \in k \right\}$$
   be the ring of 2×2 upper triangular matrices. Show that R is not local and that R has no nontrivial central idempotents. In particular, if k is finite, this gives the desired example.

(4) Let k be a field. Consider the polynomial ring $R = k[x, y, z, w]$ as a graded ring in the usual way. Compute the Hilbert polynomial and Hilbert series of the quotient ring R/I where $I = (x, y) \cap (z, w)$.

(5) Corollary 20.8 says that if R is a polynomial ring where each variable has degree 1, then the Hilbert function of a finitely generated module M is eventually a polynomial function. Here's the general setting.

A **quasi-polynomial** is a function $f\colon \mathbf{Z} \to \mathbf{Q}$ which is of the form $n \mapsto \sum_{i=0}^{d} c_i(n)n^i$ where $c_i(n)\colon \mathbf{Z} \to \mathbf{Q}$ is a periodic function for each i, i.e., there exists p_i such that $c_i(n) = c_i(n + p_i)$ for all n; if p_i is as small as possible, this is the minimal period of c_i.

In the notation of Theorem 20.7, show that there exists a quasi-polynomial $h(M, n)$ such that the Hilbert function of M is given by $h(M, n)$ for sufficiently large n, and the minimal period of each coefficient function divides $k_1k_2 \cdots k_r$.

2. Suggested exercises (don't submit)

From Altman–Kleiman:

- Chapter 18: 17, 26, 27
- Chapter 19: 2, 5, 13, 16
- Chapter 20: 5, 6, 9