Math 742, Spring 2016
Homework 4
Due: February 19

1. Exercises

(1) Let R be an integral domain and let $R[x]$ be the polynomial ring in one variable. Given $f \in R[x]$, the **content** of f, denoted $\text{cont}(f)$, is the ideal of R generated by the coefficients of f.

(a) Given $f, g \in R[x]$, show that
$$\text{cont}(fg) \subseteq \text{cont}(f)\text{cont}(g) \subseteq \sqrt{\text{cont}(fg)}.$$

(b) Pick $a, b, c, d \in R[x]$ and assume that $ab = cd$. Show that if $p \in R$ is a prime element that divides a, then p divides either c or d.

(c) Now assume R is a unique factorization domain. Prove **Gauss' Lemma**: Let K be the fraction field of R. Show that if $f \in R[x]$ is irreducible, then f is also irreducible in the larger ring $K[x]$.

(2) Let n be a square-free integer (i.e., every prime divides n at most once). Let $\mathbf{Q}$ be the rational numbers. Define
$$\mathbf{Q}(\sqrt{n}) = \{a + b\sqrt{n} \mid a, b \in \mathbf{Q}\}.$$

(a) Verify that $\mathbf{Q}(\sqrt{n})$ is a field.

(b) If $b \neq 0$, show that $a + b\sqrt{n}$ satisfies a unique monic degree 2 polynomial with rational coefficients.

(c) Determine the integral closure of $\mathbf{Z}$ in $\mathbf{Q}(\sqrt{n})$.

(d) What happens if we don't assume n is square-free?

(3) Let M be an R-module. Define the **support** of M to be
$$\text{Supp}(M) = \{P \in \text{Spec}(R) \mid M_P \neq 0\}.$$

(a) Show that $\text{Supp}(M) \subseteq V(\text{Ann}(M))$, and that equality holds if M is finitely generated.

(b) Give an example where $\text{Supp}(M)$ is not a closed subset of $\text{Spec}(R)$ (and in particular is not equal to $V(\text{Ann}(M))$).

(c) Let N be another R-module. Show that $\text{Supp}(M \otimes_R N) \subseteq \text{Supp}(M) \cap \text{Supp}(N)$, and that equality holds if M and N are finitely generated.

(4) Let $0 \to A \to B \to C \to 0$ be a sequence of R-modules. Show that the following are equivalent:

(a) $0 \to A \to B \to C \to 0$ is exact.

(b) $0 \to A_P \to B_P \to C_P \to 0$ is exact for all prime ideals P.

(c) $0 \to A_P \to B_P \to C_P \to 0$ is exact for all maximal ideals P.

(5) In this exercise, we'll explore localization in the noncommutative setting. So, in this exercise, R denotes a not necessarily commutative ring, i.e., we have all of the axioms for a ring except $ab = ba$ is no longer required.

(a) Given a multiplicative subset $S \subseteq R$, call a ring homomorphism $f \colon R \to R'$ S-**inverting** if $f(s)$ is a unit for all $s \in S$. Show that there exists a ring R_S, along with an S-inverting map $\phi \colon R \to R_S$ which is universal in the sense that for any other S-inverting map $f \colon R \to R'$, there exists a unique $g \colon R_S \to R'$ such that $f = g \circ \phi$. If R is commutative, show that $R_S = S^{-1}R$.

(b) In the commutative setting, we can construct R_S using "fractions", but this might not be possible in general: let k be a (commutative) field and let $R = k\langle X, Y\rangle$ be the ring of noncommutative polynomials[1]. Describe the ring R_S where S is the multiplicative subset generated by $\{X, Y\}$.

(c) Exercise 11.2 of Altman–Kleiman says that, if R is commutative, then $S^{-1}R = 0$ if and only if S contains a nilpotent element. This can also fail in the noncommutative setting: Let k be a field and let $n \geq 2$ be an integer. Set $R = M_n(k)$ to be the ring of $n \times n$ matrices with entries in k. For $1 \leq i, j \leq n$, let E_{ij} be the matrix[2] with a 1 in the (i, j) position and 0's elsewhere. Show that $R_S = 0$ where $S = \{E_{1,1}\}$.

2. Suggested exercises (don't submit)

From Altman–Kleiman:

- Chapter 10: 22, 35
- Chapter 11: 2, 8, 18, 25, 32
- Chapter 12: 6, 8, 14, 28

3. Further reading

The issues that come up in Exercise 5 illustrate that localization for noncommutative rings can be subtle. See Chapter 4 of T.-Y. Lam, *Lectures on Modules and Rings* for more information on noncommutative localization.

[1]A noncommutative monomial is a sequence of X's and Y's and we take the product by concatenating them, e.g., $(X^3YX^2Y^5)(YX^2) = X^3YX^2Y^6X^2$. X and Y do not commute, but they do commute with the elements of k, and a noncommutative polynomial is a finite linear combination of noncommutative monomials with coefficients in k.

[2]These are called matrix units.