Math 742, Spring 2016
Homework 2
Due: February 5

1. Exercises

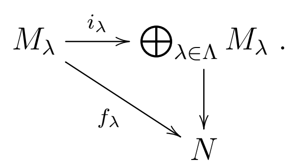
(1) Let R be a ring and let $0 \to M_1 \to M_2 \to M_3 \to 0$ be a short exact sequence of R-modules. Show that M_2 is finitely generated if M_1 and M_3 are finitely generated.

Furthermore, show that if M_2 is finitely generated, then M_3 is finitely generated. Give an example where M_2 is finitely generated but M_1 is not.

(2) Let R be a ring and let $\mathcal{C}$ be the category of R-modules. Let Λ be a set and suppose we are given modules indexed by Λ: $\{M_\lambda\}_{\lambda \in \Lambda}$.

(a) Show that the assignment $N \mapsto \prod_{\lambda \in \Lambda} \operatorname{Hom}_R(M_\lambda, N)$ defines a functor $\mathcal{C} \to \mathsf{Set}$. Show that it is representable, i.e., isomorphic to h^M where $M = \bigoplus_{\lambda \in \Lambda} M_\lambda$.

(b) Consider the inclusions $i_\lambda \colon M_\lambda \to M$ which sends $m \in M_\lambda$ to the sequence which is m in position λ and 0 elsewhere. Show that this satisfies the following universal mapping property: given any other module N and homomorphisms $f_\lambda \colon M_\lambda \to N$, there is a *unique* homomorphism $M \to N$ such that the following diagram commutes for all $\lambda \in \Lambda$:



If we're given another module P and homomorphisms $j_\lambda \colon M_\lambda \to P$ which satisfy the same universal mapping property, then show, just using the definition, that M and P are canonically isomorphic. Reinterpret (a) using Yoneda's lemma to give another proof of this fact.

A special case is when $M_\lambda = R$ for all $\lambda \in \Lambda$, in which case we get the universal mapping property for a free module $R^{\oplus \Lambda}$.

(c) Similarly, show that the assignment $N \mapsto \prod_{\lambda \in \Lambda} \operatorname{Hom}_R(N, M_\lambda)$ defines a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. Show that it is representable, i.e., isomorphic to $h_{M'}$ where $M' = \prod_{\lambda \in \Lambda} M_\lambda$. Translate this into a universal mapping property for the direct product as we did for the direct sum (you don't have to reprove anything).

(3) Let R be a ring and let m, n, p be positive integers.

(a) Show that a homomorphism between free R-modules $R^n \to R^m$ is the same thing as an $m \times n$ matrix whose entries come from R, and that composition $R^n \to R^m \to R^p$ can be computed using matrix multiplication.

In particular, an endomorphism $\alpha \colon R^n \to R^n$ is the same as a square matrix, so we can define its determinant in the usual way:

$$\det(\alpha) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \alpha_{1,\sigma(1)} \cdots \alpha_{n,\sigma(n)}.$$

It satisfies all of the familiar properties from linear algebra: it's independent of choice of basis for R^n, there's a Laplace expansion formula, and $\det(\alpha\beta) = \det(\alpha)\det(\beta)$. (You don't need to prove this.)

(b) Show that α is bijective if and only if $\det(\alpha)$ is a unit.

(c) Show that α is injective if and only if $\det(\alpha)$ is not a zerodivisor.
(d) In particular, injective need not imply bijective. However, show that if α is surjective, then α is bijective.

(4) Let R be a principal ideal domain. Here are two facts you may use without proof:
- A submodule of a finitely generated R-module is also finitely generated[1].
- (Smith normal forms): Given an $n \times m$ matrix A with entries in R, there exists an $n \times n$ matrix S and an $m \times m$ matrix T such that SAT is diagonal (i.e., only has entries in the (i, i) positions), and each element is of the form up^d where u is a unit, p is a prime element, and $d \geq 0$ is an integer.

Use them to show the following:
(a) Show that every finitely generated R-module M is isomorphic to the direct sum of a free R-module and its torsion submodule[2] and that, furthermore, the torsion submodule is isomorphic to a direct sum of modules $R/(p^d)$ where p is prime and $d > 0$ is an integer.
(b) Let k be a field and let X be an $n \times n$ matrix. Describe a $k[x]$-module structure on k^n where x acts by X. Show that this is a finitely generated torsion module.
Now assume k is algebraically closed. Interpret the decomposition for the module M from (a) in terms of a normal form for X (this is the **Jordan canonical form**).

(5) Let R be a ring and suppose we have the following commutative diagram of R-modules:

$$\begin{array}{ccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle i} \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' \end{array}$$

Assume the top row and bottom row are both exact.
(a) If g, i are injective and f is surjective, show that h is injective.
(b) If f, h are surjective and i is injective, show that g is surjective.

2. Suggested exercises (don't submit)

From Altman–Kleiman:
- Chapter 4: 3, 17, 18, 19
- Chapter 5: 14, 16, 22, 29
- Chapter 6: 5, 9
- Chapter 7: 2, 3, 9

[1]This is better studied in the general context of noetherian rings, which comes later in the course.
[2]m is in the torsion submodule if there exists nonzero $x \in R$ such that $xm = 0$