Math 742, Spring 2016
Homework 12
Due: May 2

1. Exercises

Given a subset $I \subset \{1, \dots, n\}$, let $i_1 < \cdots < i_m$ be the elements of I and let $j_1 < \cdots < j_{n-m}$ be the elements in the complement of I. Let $s(I)$ denote the sign of the permutation that takes $i_1, \dots, i_m, j_1, \dots, j_{n-m}$ to $1, 2, \dots, n$.

(1) Let E be a finite dimensional k-vector space with bilinear form $\beta \colon E \times E \to k$. Recall that if we pick a basis $x_1, \dots, x_n$ for E, we get a $n \times n$ matrix representation M of β by defining the (i, j) entry to be $\beta(x_i, x_j)$. Let $y_1, \dots, y_n$ be a different basis and let N be the matrix representation in this basis. Show that $M = CNC^T$ for some matrix C (in the process you should explain what C is).

(2) Let $A = (a_{ij})$ be an $n \times n$ matrix. Pick $1 \le m < n$. Given a subset $I \subset \{1, \dots, n\}$ of size m with $i_1 < \cdots < i_m$, let A_I be the $m \times m$ submatrix of A with rows indexed by I and columns $1, \dots, m$ (in the same order as they appear in A). Let B_I be the $(n-m) \times (n-m)$ submatrix of A with rows indexed by the complement of I and columns $m+1, \dots, n$ (again, in the same order as they appear in A). Use the exterior algebra interpretation of the determinant to show that[1]

$$\det(A) = \sum_{I \subset \{1, \dots, n\}} s(I) \det(A_I) \det(B_I)$$

where the sum is over all subsets I of size m.

(3) Let E be a vector space of dimension $2n$ over k. Let A be a matrix representation of f with respect to a basis $e_1, \dots, e_{2n}$.

As we said in class, an alternating form $f \colon E \times E \to k$ can naturally be interpreted as an element $\omega \in \bigwedge^2 E$ where $\omega = \sum_{1 \le i < j \le 2n} f(e_i, e_j) e_i \wedge e_j$. Show that

$$\omega^n = n! \operatorname{Pf}(A) e_1 \wedge e_2 \wedge \cdots \wedge e_{2n}.$$

[Hint: Both sides of the formula transform in a predictable way if you do a change of basis, so figure out how that works and then pick an easy basis to work with.]

(4) Let $\phi \colon E \to E$ be a linear operator where E is a vector space of dimension n.
   (a) Let $\lambda_1, \dots, \lambda_n$ be its eigenvalues (with multiplicities). Show that the eigenvalues of $\bigwedge^r \phi \colon \bigwedge^r E \to \bigwedge^r E$ (with multiplicities) are the products $\lambda_{i_1} \lambda_{i_2} \cdots \lambda_{i_r}$ with $i_1 < i_2 < \cdots < i_r$.
   (b) Let $x^n + a_{n-1} x^{n-1} + \cdots + a_0 = \det(xI_n - \phi)$ be the characteristic polynomial of ϕ. Conclude that $(-1)^r a_{n-r} = \operatorname{trace}(\bigwedge^r \phi)$.

2. Further reading

- There is a lot of material in Lang, Chapter XV about the structure of bilinear forms. In particular, I assume you saw the spectral theorem (this says that Hermitian matrices over **C** and symmetric matrices over **R** are diagonalizable) in an undergraduate linear algebra course. We won't use it in this course, but you should know it. If not, please read those sections. It is probably also relevant to understand some of the basic structure theory of symmetric bilinear forms: they are more complicated than

[1] This is sometimes called the generalized Laplace expansion formula since $m = 1$ is the usual one.

alternating bilinear forms since the properties of the field matters (whereas it made no difference in the alternating case which field we studied).

- Pfaffians have a lot in common with determinants. For example, they have Laplace expansion formulas and also can be written as a sum over permutations (more precisely, over certain kinds of permutations). See the wikipedia page for some of these formulas:

  `https://en.wikipedia.org/wiki/Pfaffian`

  I chose not to include them on the homework since that would probably make the assignment too long.