Math 742, Spring 2016
Homework 1
Due: January 29

1. Notes and conventions

- "Ring" means "commutative ring" with multiplicative unit 1. Ring homomorphisms $f\colon R \to S$ always satisfy $f(1) = 1$. Prime ideals do not contain the element 1. The zero ring (where the only element is $0 = 1$) is a ring by our definition.
- $\mathbf{Z}$ is the ring of integers.
- If you aren't familiar with categories, functors, natural transformations, you can find their definitions in Lang, §I.11 (or wikipedia). Also, don't worry about set-theoretic issues with categories: there will be time for that later in life.
- Set is the category of sets and functions.

2. Exercises

(1) Let A be a ring and let $A[x]$ be the polynomial ring in one variable over A. Pick $f = a_0 + a_1x + \cdots + a_nx^n \in A[x]$. Show that f is a unit in $A[x]$ if and only if a_0 is a unit in A and $a_1, a_2, \ldots, a_n$ are nilpotent.

Deduce that if $a \in A$ is nilpotent, then $1 + a$ is a unit in A.

(2) Let A be a ring and let I, J be ideals of A. Let $\sqrt{I}$ be the radical of I. Show that

$$\sqrt{IJ} = \sqrt{I \cap J} = \sqrt{I} \cap \sqrt{J}$$

(IJ means the product of I and J).

(3) Let A be a ring and $\mathfrak{N}$ its nilradical. Show that the following are equivalent:
  - (a) A has exactly one prime ideal,
  - (b) every element of A is either a unit or nilpotent,
  - (c) $A/\mathfrak{N}$ is a field.

(4) Let A be a ring and let $\operatorname{Spec}(A)$ denote the set of all prime ideals of A. Given a subset $S \subset A$, let $V(S)$ be the set of prime ideals containing S.
  - (a) If I is the ideal generated by S, show that $V(S) = V(I) = V(\sqrt{I})$.
  - (b) Show that $\operatorname{Spec}(A)$ is a topological space if we define the closed sets to be the subsets of the form $V(S)$ as S varies over all subsets (equivalently, over all ideals) of A. This is the **Zariski topology** on $\operatorname{Spec}(A)$.
  - (c) Given a homomorphism $\phi\colon A \to B$, the map $P \mapsto \phi^{-1}(P)$ defines a function $\phi^*\colon \operatorname{Spec}(B) \to \operatorname{Spec}(A)$. Show that ϕ^* is continuous and that Spec is a contravariant functor from the category of rings to the category of topological spaces.
  - (d) Describe the topological space $\operatorname{Spec}(\mathbf{Z})$.

(5) Given $f \in A$, let $D(f) \subseteq \operatorname{Spec}(A)$ be the set of prime ideals that do *not* contain f.
  - (a) Show that $D(f)$ is open and form a basis of open sets[1] for the Zariski topology.
  - (b) Show that $D(fg) = D(f) \cap D(g)$.
  - (c) Show that $D(f) = D(g)$ if and only if $\sqrt{(f)} = \sqrt{(g)}$. In particular, $D(1) = \operatorname{Spec}(A)$ and $D(f) = \emptyset$ if f is nilpotent.

[1] A collection B of open sets is a basis if every open set can be written as a union of open sets in B.

(d) Show that $\mathrm{Spec}(A)$ is quasi-compact[2]. More generally, show that $D(f)$ is quasi-compact.

(e) Given a homomorphism $\phi\colon A \to B$ and $f \in A$, show that $(\phi^*)^{-1}D(f) = D(\phi(f))$.

(6) Let $\mathcal{C}, \mathcal{D}$ be categories. Let $[\mathcal{C}, \mathcal{D}]$ denote the category whose objects are functors $F\colon \mathcal{C} \to \mathcal{D}$ and where morphisms $F \to G$ are natural transformations. Let $\mathcal{C}^{\mathrm{op}}$ be the opposite category of $\mathcal{C}$ (each morphism $f\colon X \to X'$ is replaced by a morphism $f^{\mathrm{op}}\colon X' \to X$).

Note that, given a functor $F\colon \mathcal{C} \to \mathcal{D}$, and objects $X, X' \in \mathcal{C}$, one has an induced function $\mathrm{Hom}_{\mathcal{C}}(X, X') \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(X'))$. A functor F is **full** if this map is always surjective, and is **faithful** if it is always injective (and **fully faithful** if both).

Define a functor

$$h\colon \mathcal{C} \to [\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$$

which sends an object $X \in \mathcal{C}$ to the contravariant functor $Y \mapsto \mathrm{Hom}_{\mathcal{C}}(Y, X)$ and a morphism $F\colon X \to X'$ to the post-composition map $\mathrm{Hom}_{\mathcal{C}}(Y, X) \to \mathrm{Hom}_{\mathcal{C}}(Y, X')$. Instead of writing $h(X)$, we will write h_X.

(a) Show that a fully faithful functor $F\colon \mathcal{C} \to \mathcal{D}$ is injective on isomorphism classes of objects, i.e., if $F(X) \cong F(X')$ then $X \cong X'$.

(b) Prove **Yoneda's lemma**: h is a fully faithful functor.
(You'll probably want to show that given a functor $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$ and an object $X \in \mathcal{C}$, the set of natural transformations $h_X \to F$ is naturally isomorphic to $F(X)$.)

(c) Formulate and prove a covariant version of Yoneda's lemma which uses $h^X = \mathrm{Hom}_{\mathcal{C}}(X, Y)$ in place of h_X.

Remark. Objects in the image of h in $[\mathcal{C}^{\mathrm{op}}, \mathrm{Set}]$ are called **representable functors**. This exercise shows that objects of $\mathcal{C}$ can be replaced by their representable functors without loss of information. This perspective can be used to define certain operations, like tensor products, by focusing more on the behavior of the representable functor. This is closely related to the idea of universal properties, which you have seen in Math 741.

(7) Let $\mathcal{C}$ be the category of rings. Show that the following two functors $\mathcal{C} \to \mathrm{Set}$ are representable, i.e., isomorphic to functors of the form h^R for some ring R.

(a) The functor that sends each ring to a one-point set $\{*\}$ and each homomorphism to the identity map of this set.

(b) The functor that sends each ring to its set of elements (the forgetful functor) and each homomorphism to the corresponding function.

3. Suggested exercises (don't submit)

There are plenty of exercises in Altman–Kleiman. At the very least, you should read the statement of each to familiarize yourself with basic facts. There are solutions in the back for all of them, so feel free to read them. Here are some which might be good to spend more time thinking about (but don't turn these in):

- Chapter 1: 5, 14, 16
- Chapter 2: 11, 12, 16, 22
- Chapter 3: 23, 25, 26

Exercises 27 and 28 in Chapter 1 of Atiyah–Macdonald are also good to study.

[2]Quasi-compact means that every open covering has a finite subcovering. This is what you might have called "compact" before, but that usually gets reserved for Hausdorff spaces, which $\mathrm{Spec}(A)$ almost never is.