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Convergence tests for series

Here's a few useful tests for when a series $\sum_{k=1}^{\infty} a_k$ converges.

1. Alternating series test

Theorem 1 (Alternating series test)**.** *Assume that a_n are all non-negative, that $a_1 \geq a_2 \geq a_3 \geq \cdots$, and that $\lim_{n\to\infty} a_n = 0$. Then $\sum_{k=1}^{\infty}(-1)^k a_k$ converges.*

Example 1. Set $a_n = 1/n$. Then all a_n are non-negative, $1 \geq 1/2 \geq 1/3 \geq \cdots$, and $\lim_{n\to\infty} 1/n = 0$. So $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges.

2. Integral comparison test

Theorem 2 (Integral comparison test)**.** *Let $f(x)$ be a* **decreasing, non-negative** *continuous function for $x \geq 1$. Then $\sum_{k=1}^{\infty} f(k)$ converges if and only if $\int_1^{\infty} f(x)dx$ converges.*

Example 2.

- $\frac{1}{x}$ is a decreasing and non-negative function for $x \geq 1$. So $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges because $\int_1^{\infty} \frac{dx}{x}$ diverges by p-test. This is called the **harmonic series**.
- Similarly, $\frac{1}{x^2}$ is also a decreasing and non-negative function for $x \geq 1$. So $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges because $\int_1^{\infty} \frac{dx}{x^2}$ converges (again by p-test).
- More generally, $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges if and only if $p > 1$.

Remark 1. Let me point out one thing: even though the convergence property of the series $\sum_{k=1}^{\infty} f(k)$ and the convergence property of the integral $\int_1^{\infty} f(x)dx$ are tied to each other, the values, if they converge, can be very different. In fact, the integral is usually easier to evaluate!

For example, when $f(x) = 1/x^2$,

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{M\to\infty} \left.\frac{-1}{x}\right|_1^M = 1,$$

but

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \approx 1.6449.$$

I don't know of any derivations of this identity that are simple enough to include here unfortunately.

3. Comparison test

Theorem 3. *If $b_k \geq |a_k|$ for all k and $\sum_{k=1}^{\infty} b_k$ converges, then so does $\sum_{k=1}^{\infty} a_k$.*

A special case is when $b_k = |a_k|$. If $\sum_{k=1}^{\infty} |a_k|$ converges, then the series is said to be **absolutely convergent**. So the theorem says that an absolutely convergent series is also convergent. A convergent series might not be absolutely convergent though. For example, as we saw above, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges but $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges. A convergent series which is not absolutely convergent is called **conditionally convergent**.

4. Limit comparison test

Theorem 4 (Limit comparison test for series)**.** *If $a_n > 0$ and $b_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 < L < \infty$, then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.*

Example 3.

- Does $\sum_{k=1}^{\infty} \frac{1}{k+1}$ converge? Take $a_n = \frac{1}{n+1}$ and $b_n = \frac{1}{n}$. Then
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n+1} = 1.$$
Since $\sum_{k=1}^{\infty} \frac{1}{k}$ is the harmonic series and diverges, our original series also diverges.
- In a similar way, you can show that $\sum_{k=1}^{\infty} \frac{1}{k^2+1}$ converges since $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges.

5. Ratio test

Theorem 5 (Ratio test)**.** *Let $(a_n)_{n=1}^{\infty}$ be a sequence. Assume that $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$.*

- *If $L > 1$, the series $\sum_{k=1}^{\infty} a_k$ diverges.*
- *If $L < 1$, the series $\sum_{k=1}^{\infty} a_k$ converges.*
- *If $L = 1$, the ratio test tells you nothing.*

Example 4.

- Consider $\sum_{k=1}^{\infty} 2^k$. Set $a_k = 2^k$. Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} 2 = 2 > 1,$$
so the series diverges.

- Consider $\sum_{k=1}^{\infty} \frac{1}{3^k}$. Set $a_k = \frac{1}{3^k}$. Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{1}{3} = \frac{1}{3} < 1,$$
so the series converges.
- Why is the test inconclusive if $L = 1$? Consider two examples. One is the series $\sum_{k=1}^{\infty} 1$ which has $L = 1$ and obviously diverges. Another is the series $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Then with $a_k = 1/k^2$, we get
$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = 1,$$
but this series converges as we saw before.

6. Taylor series

The ratio test is very helpful for determining convergence of Taylor series. Let's work through the two examples in the book and then we'll do another one after.

Example 5.
- Consider the geometric series $\sum_{k=0}^{\infty} x^k$. For which x does it converge? Use the ratio test with $a_k = x^k$. Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} |x| = |x|.$$
So the ratio test tells us that if $|x| < 1$, the geometric series converges, and if $|x| > 1$, the geometric series diverges. How about $|x| = 1$? There are two possibilities, $x = 1$ and $x = -1$. It's clear that $\sum_{k=0}^{\infty} 1$ diverges. How about $\sum_{k=0}^{\infty} (-1)^k$? The partial sums are $1, 0, 1, 0, 1, 0, \dots$ which doesn't converge either.
  Conclusion: $\sum_{k=1}^{\infty} x^k$ converges if $|x| < 1$ and diverges if $|x| \geq 1$.
- Consider the Taylor series $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ for e^x. For which x does it converge? Again, use the ratio test with $a_k = x^k/k!$. Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|}{n+1} = 0.$$
So $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!}$ converges for all x by the ratio test.

One more example:

Example 6. For which x does $\displaystyle\sum_{k=1}^{\infty} \frac{(-3)^k x^k}{\sqrt{k}}$ converge? Again, we'll use the ratio test with $a_k = \dfrac{(-3)^k x^k}{\sqrt{k}}$. Then
$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|-3x\sqrt{\frac{n}{n+1}}\right| = 3|x| \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = 3|x|.$$

For the last equality, note that $\lim_{n\to\infty} \frac{n}{n+1} = 1$ and then use that $\sqrt{\lim_{n\to\infty} \frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}}$ since $\sqrt{x}$ is continuous at $x = 1$.

In particular, the ratio test tells us that the series converges for $|x| < 1/3$ and diverges for $|x| > 1/3$. What's left is to check $x = 1/3$ and $x = -1/3$.

When $x = 1/3$, we get the series $\sum_{k=1}^{\infty} \frac{(-1)^k}{\sqrt{k}}$. This converges by the alternating series test (check this). When $x = -1/3$, we get the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. This diverges by the integral test because $\int_1^{\infty} \frac{dx}{x^{1/2}}$ diverges.

In conclusion, the series converges if $-\frac{1}{3} < x \leq \frac{1}{3}$ and otherwise it diverges.

7. Convergence of Taylor series

If the Taylor series converges, you can ask what the limit is. A natural guess is the original function. How might you check this? If $f(x)$ is your function and its Taylor series is $\sum_{k=0}^{\infty} a_k x^k$, then to say that it converges to $f(x)$, you want that

$$\lim_{n\to\infty} \left(f(x) - \sum_{k=0}^{n} a_k x^k \right) = 0.$$

The left side is familiar: it's the remainder $R_n f(x)$. So $T_\infty f(x) = f(x)$ if $\lim_{n\to\infty} R_n f(x) = 0$.

The book shows that this is true for $f(x) = \frac{1}{1-x}$ and $f(x) = e^x$. Let's look at some more examples.

Example 7. Consider $f(x) = \sin x$. We don't have a good formula for $R_n f(x)$, but we can bound it by Taylor's inequality. To do this, we need to find a bound M for $|f^{(n+1)}(x)|$. Depending on n, this is one of $\sin x, \cos x, -\sin x, -\cos x$. In all cases, we can take $M = 1$. So Taylor's inequality tells us

$$|R_n f(x)| \leq \frac{|x|^{n+1}}{(n+1)!}.$$

As we saw in class (also in the book), $\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0$ for any x, so $\lim_{n\to\infty} R_n f(x) = 0$ by the sandwich theorem. So we can write

$$\sin x = T_\infty \sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}.$$

Once you have this, you're free to plug in whatever values of x you like (since we know the convergence for all x). For example:

$$\sin(1) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} = 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots.$$

In fact, almost all functions you've seen in this class have the property that their Taylor series converge to it. Here's one warning example though.

Example 8. Consider

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}.$$

In HW8, you showed that $f(x)$ is $o(x^n)$ for all n. This means that $T_n f(x) = 0$ for all n (you have to check that $f(x)$ can be differentiated n times for the Taylor polynomial to make sense, but let's not worry about that right now). So $T_\infty f(x) = 0$, which just converges to 0. But $f(x)$ is not the 0 function, so $f(x) \neq T_\infty f(x)$ in this case.

Here's a different warning example:

Example 9. Let $f(x) = \frac{1}{1-x}$. In the book, it's shown that $f(x) = T_\infty f(x) = \sum_{k=0}^\infty x^k$ for $|x| < 1$. So for example, plugging in $x = 1/2$, you get

$$2 = \frac{1}{1 - \frac{1}{2}} = \sum_{k=0}^{\infty} \frac{1}{2^k},$$

which we've already seen before. But we have to make sure we only plug in values x with $|x| < 1$. For example, if you plug in $x = 2$, you get $f(2) = -1$ on one side and $\sum_{k=0}^\infty 2^k$ on the other side, but this series diverges, so it's not a valid identity.