Recurrence relations
Math 475
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Say we have a sequence of numbers $a_0, a_1, a_2, \dots$ that satisfies a recurrence relation of the form

$$a_n = Ca_{n-1} + Da_{n-2}$$

whenever $n \ge 2$ (here C, D are some constants and $D \ne 0$). We want to find a closed formula for a_n.

The **characteristic polynomial** of this recurrence relation is defined to be

$$t^2 - Ct - D.$$

The roots of this polynomial are $\dfrac{C \pm \sqrt{C^2 + 4D}}{2}$. Call them α and β. So we can factor the characteristic polynomial as

$$t^2 - Ct - D = (t - \alpha)(t - \beta). \tag{1}$$

Comparing constant terms, we get $\alpha\beta = D$, so $\alpha \ne 0$ and $\beta \ne 0$ because we assumed that $D \ne 0$.

Here is the first statement:

Theorem 1. *If $\alpha \ne \beta$, then there are constants c_0 and c_1 such that*

$$a_n = c_0\alpha^n + c_1\beta^n$$

for all n.

To solve for the coefficients, plug in $n = 0$ and $n = 1$ to get

$$a_0 = c_0 + c_1$$
$$a_1 = \alpha c_0 + \beta c_1.$$

Then you have to solve for c_0, c_1 (a_0, a_1 are part of the original sequence, so are given to you).

Proof of Theorem 1. Define a generating function

$$A(x) = \sum_{n=0}^{\infty} a_n x^n.$$

The recurrence relation says that we have a relation of the form

$$A(x) = CxA(x) + Dx^2A(x) + a_0 + (a_1 - Ca_0)x.$$

We can rewrite this as

$$A(x) = \frac{a_0 + (a_1 - Ca_0)x}{1 - Cx - Dx^2}. \tag{2}$$

We want to factor the denominator. To do this, plug in $t \mapsto x^{-1}$ into (1) and multiply by x^2 to get

$$1 - Cx - Dx^2 = (1 - \alpha x)(1 - \beta x).$$

Now we can apply partial fraction decomposition to (2) to write

$$A(x) = \frac{c_0}{1 - \alpha x} + \frac{c_1}{1 - \beta x}$$

for some constants c_0, c_1. But these terms are both geometric series, so we can further write

$$A(x) = c_0 \sum_{n=0}^{\infty} \alpha^n x^n + c_1 \sum_{n=0}^{\infty} \beta^n x^n.$$

The coefficient of x^n on the left side is a_n and the coefficient of x^n on the right side is $c_0\alpha^n + c_1\beta^n$. So we have equality for all n. □

There is a loose end: what if $\alpha = \beta$?

Theorem 2. *If $\alpha = \beta$, then there are constants c_0 and c_1 such that*

$$a_n = c_0\alpha^n + c_1 n\alpha^n$$

for all n.

Again, to solve for c_0, c_1, just plug in $n = 0, 1$ to get a system of equations:

$$\begin{aligned} a_0 &= c_0 \\ a_1 &= c_0\alpha + c_1\alpha. \end{aligned}$$

Proof. We can start in the same way as in the previous proof. The only difference is that we are trying to take the partial fraction decomposition of

$$A(x) = \frac{a_0 + (a_1 - Ca_0)x}{(1-\alpha x)^2}.$$

This can still be done, but now it looks like

$$\frac{d_0}{1-\alpha x} + \frac{d_1}{(1-\alpha x)^2}$$

for some constants d_0, d_1. The first is a geometric series, and the second we've seen: remember that $1/(1-x)^2 = \sum_{n\geq 0}(n+1)x^n$. So we get instead

$$A(x) = d_0 \sum_{n=0}^{\infty} \alpha^n x^n + d_1 \sum_{n=0} (n+1)\alpha^n x^n.$$

Comparing coefficients, we get

$$a_n = d_0\alpha^n + d_1(n+1)\alpha^n = (d_0 + d_1)\alpha^n + d_1 n\alpha^n.$$

So $c_0 = d_0 + d_1$ and $c_1 = d_1$. □

Let's finish with the example of the Fibonacci numbers f_n. These are defined by

$$\begin{aligned} f_0 &= 1 \\ f_1 &= 1 \\ f_n &= f_{n-1} + f_{n-2} \qquad \text{for } n \geq 2 \end{aligned}$$

So the characteristic polynomial is $t^2 - t - 1$. Its roots are $\dfrac{1 \pm \sqrt{5}}{2}$. Set $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. So we have

$$f_n = c_0\alpha^n + c_1\beta^n$$

and we have to solve for c_0 and c_1. Plug in $n = 0, 1$ to get:

$$\begin{aligned} 1 &= c_0 + c_1 \\ 1 &= c_0\alpha + c_1\beta. \end{aligned}$$

So $c_0 = 1 - c_1$; plug this into the second formula to get $1 = (1 - c_1)\alpha + c_1\beta$. Rewrite this as $1 - \alpha = c_1(\beta - \alpha)$. We can simplify this: $\beta - \alpha = -\sqrt{5}$ and $1 - \alpha = (1 - \sqrt{5})/2$. So

$$c_1 = -\frac{1 - \sqrt{5}}{2\sqrt{5}}, \qquad c_0 = 1 - c_1 = \frac{1 + \sqrt{5}}{2\sqrt{5}}.$$

In conclusion:

$$\begin{aligned} f_n &= \frac{1 + \sqrt{5}}{2\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^n - \frac{1 - \sqrt{5}}{2\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}} \left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1 - \sqrt{5}}{2}\right)^{n+1}. \end{aligned}$$

(The last step wasn't necessary, we just did that to reduce the number of radical signs.)

What about higher degree recurrence relations like

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k} \qquad \text{for } n \geq k?$$

This can be solved in the same way: one has to first find the roots of the characteristic polynomial $t^k - C_1 t^{k-1} - C_2 t^{k-2} - \cdots - C_k$ and apply partial fraction decomposition. The simplest case is when the roots $\alpha_1, \ldots, \alpha_k$ are all distinct. In this case, we can say that there exist constants $c_1, \ldots, c_k$ such that

$$a_n = c_1 \alpha_1^n + \cdots + c_k \alpha_k^n$$

for all n. In order to solve for $c_1, \ldots, c_k$, we have to consider $n = 0, \ldots, k - 1$ separately to get a system of k linear equations in k variables. When the roots appear with multiplicities, we have to do something like we did in Theorem 2. For example, if $k = 5$ and the roots are α with multiplicity 3 and β with multiplicity 2 (and $\alpha \neq \beta$), then we would have

$$a_n = c_1 \alpha^n + c_2 n \alpha^n + c_3 n^2 \alpha^n + c_4 \beta^n + c_5 n \beta^n.$$

This should look familiar to you if you've ever solved a linear homogeneous differential equation with constant coefficients.

I'll leave it to you to formulate the general case, though we won't be doing anything more with it in this class.