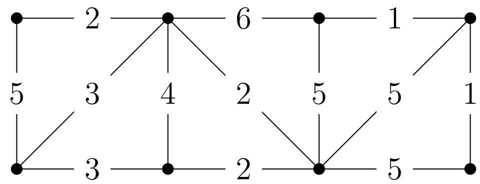
Math 475, Fall 2015
Homework 8
Due: Friday, Nov. 6

(1) Up to isomorphism, there are 6 different trees with 6 vertices. Draw all of them. Justify why they aren't isomorphic to each other (try to find a unique property that each one has that the others don't).

(2) Find a minimum weight spanning tree in the following graph. What is its weight?



(3) In this exercise, we will give a different proof of Cayley's formula that the number of trees on vertex set $[n]$ is n^{n-2}.

Let $x_1, \dots, x_n$ be variables. Given a tree T on vertex set $[n]$, let d_i be the degree of the vertex i. Define $x(T) = x_1^{d_1} x_2^{d_2} \cdots x_n^{d_n}$. For example when $n = 3$:

T	2 — 1 — 3	1 — 2 — 3	1 — 3 — 2
$x(T)$	$x_1^2 x_2 x_3$	$x_1 x_2^2 x_3$	$x_1 x_2 x_3^2$

Define $C_n(\mathbf{x}) = \sum_T x(T)$ where the sum is over all trees T on $[n]$ and define

$$D_n(\mathbf{x}) = x_1 x_2 \cdots x_n (x_1 + x_2 + \cdots + x_n)^{n-2}.$$

For example,

$$C_3(\mathbf{x}) = x_1^2 x_2 x_3 + x_1 x_2^2 x_3 + x_1 x_2 x_3^2 = x_1 x_2 x_3 (x_1 + x_2 + x_3) = D_3(\mathbf{x}).$$

The goal is to show that $C_n(\mathbf{x}) = D_n(\mathbf{x})$ for all n. We will proceed by induction on n. When $n = 1$, $C_1(\mathbf{x}) = D_1(\mathbf{x}) = 1$.

Now assume $n > 1$ and that $C_{n-1}(\mathbf{x}) = D_{n-1}(\mathbf{x})$. To show that $C_n(\mathbf{x}) = D_n(\mathbf{x})$, it is the same to show that for every monomial $x_1^{d_1} \cdots x_n^{d_n}$, its coefficient is the same in both $C_n(\mathbf{x})$ and $D_n(\mathbf{x})$.

Let $p(\mathbf{x})$ be a polynomial in the variables $x_1, \dots, x_n$ which is divisible by x_i. Define $(x_i^{-1} p(\mathbf{x}))_{x_i \mapsto 0}$ to be the result of dividing $p(\mathbf{x})$ by x_i and then plugging in $x_i = 0$.

Fact: If $d_i = 1$, then the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ in $p(\mathbf{x})$ is the same as the coefficient of $x_1^{d_1} \cdots x_{i-1}^{d_{i-1}} x_{i+1}^{d_{i+1}} \cdots x_n^{d_n}$ in $(x_i^{-1} p(\mathbf{x}))_{x_i \mapsto 0}$.

(a) Clearly $D_n(\mathbf{x})$ is divisible by x_n; explain why $C_n(\mathbf{x})$ is also divisible by x_n. Then show that

$$(x_n^{-1} D_n(\mathbf{x}))_{x_n \mapsto 0} = (x_1 + \cdots + x_{n-1}) \cdot D_{n-1}(\mathbf{x})$$
$$(x_n^{-1} C_n(\mathbf{x}))_{x_n \mapsto 0} = x_n^{-1} \widetilde{C}_n(\mathbf{x})$$

where $\widetilde{C}_n(\mathbf{x}) = \sum x(T)$ and the sum is over all trees on $[n]$ such that $\deg(n) = 1$.

(continued on next page)

(b) Show that

$$\widetilde{C}_n(\mathbf{x}) = x_n(x_1 + \cdots + x_{n-1})C_{n-1}(\mathbf{x}).$$

[**Hint:** We can interpret $x_i x_n C_{n-1}(\mathbf{x})$ where $1 \le i \le n-1$ on the right side as follows: $C_{n-1}(\mathbf{x})$ is a sum over $x(T)$ where T is a tree on $[n-1]$, and multiplying $x(T)$ by $x_i x_n$ gives $x(T')$ where T' is the tree obtained by adding the vertex n and the edge $\{i, n\}$ to T. By summing over all T and i, show that every tree on $[n]$ where n has degree 1 is obtained exactly once.]

(c) It is clear that $D_n(\mathbf{x})$ stays the same if you swap the variables x_i and x_n. Explain why the same is true for $C_n(\mathbf{x})$. Using (a), (b), and the induction hypothesis that $C_{n-1}(\mathbf{x}) = D_{n-1}(\mathbf{x})$, conclude that

$$(x_i^{-1} C_n(\mathbf{x}))_{x_i \mapsto 0} = (x_i^{-1} D_n(\mathbf{x}))_{x_i \mapsto 0}$$

for all $i = 1, \ldots, n$.

(d) In particular, (c) and the Fact above implies that the coefficients of $x_1^{d_1} \cdots x_n^{d_n}$ in both $C_n(\mathbf{x})$ and $D_n(\mathbf{x})$ are the same if there is some i such that $d_i = 1$. Explain why every monomial that has nonzero coefficient in $C_n(\mathbf{x})$ or $D_n(\mathbf{x})$ must have $d_i = 1$ for some i. Conclude that $C_n(\mathbf{x}) = D_n(\mathbf{x})$.

(e) Finally, plug in $x_1 = x_2 = \cdots = x_n = 1$ into $C_n(\mathbf{x}) = D_n(\mathbf{x})$ and deduce that there are n^{n-2} trees on the vertex set $[n]$.

Remark (i.e., the following is not an exercise): The identity $C_n(\mathbf{x}) = D_n(\mathbf{x})$ clearly contains more information than just the fact that the number of trees on $[n]$ is n^{n-2}. For example, the coefficient of $x_1^{d_1} \cdots x_n^{d_n}$ is the number of trees where $\deg(i) = d_i$, and using $D_n(\mathbf{x})$ and the multinomial theorem, we can get this number as a multinomial coefficient

$$\binom{n-2}{d_1 - 1, d_2 - 1, \ldots, d_n - 1} = \frac{(n-2)!}{(d_1-1)!(d_2-1)! \cdots (d_n-1)!}.$$

(This could also be deduced from Prüfer encoding.)