Math 475, Fall 2015
Homework 7
Due: Friday, Oct. 30

(1) Bóna 9.28: Let G be a simple graph and assume there is a walk from vertex A to vertex B. Prove that there is also a path from A to B. (Remember, a path is a walk that doesn't touch any vertex more than once.)

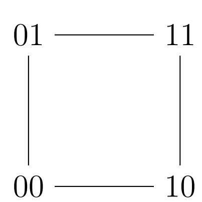
(2) Let G be a graph without loops. Show that there is always a way to put a direction on each edge of G so that there are no directed cycles (having a directed cycle means that we can start at some vertex, follow some choice of edges going along the direction, and come back to the same vertex).

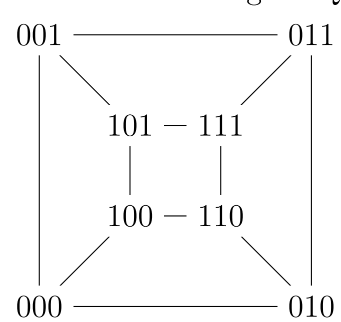
(3) If G is a simple graph with n vertices, define its **degree sequence** to be the list of the degrees $(d_1, \dots, d_n)$ of its vertices in weakly increasing order (so $d_1 \le d_2 \le \cdots \le d_n$).
  (a) Prove that if G and H are isomorphic simple graphs, then their degree sequences are the same.
  (b) Show that the converse need not be true by finding two simple graphs on 6 vertices with degree sequence $(2, 2, 2, 2, 2, 2)$ which are not isomorphic to each other.

(4) Bóna 9.34: Draw all isomorphism classes of simple graphs with 4 vertices. You may use the fact that there are 11: so just produce 11 graphs, but you should explain why your list doesn't have any repetitions.
[**Hint:** Organize them by number of edges; #3(a) might be helpful.]

(5) Let n be a positive integer. Define a simple graph Q_n as follows:
- The vertices are n-tuples $(x_1, \dots, x_n)$ with $x_i \in \{0, 1\}$ (so there are 2^n vertices).
- There is an edge between $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ if the two n-tuples agree in exactly $n - 1$ coordinates (i.e., there exists i such that $x_j = y_j$ if $j \neq i$ but $x_i \neq y_i$). So each vertex has degree n.

These are called **hypercube graphs**. Here are drawings of Q_2 and Q_3:





  (a) Bóna 9.41: Prove that if $n \ge 2$, then Q_n has a Hamiltonian cycle.
  (b) Bóna 9.43 (variant): How many Hamiltonian cycles does Q_3 have that begin and end at $(0, 0, 0)$?
  [**Hint:** Show that if we pick any path $e_1 e_2 e_3$ of length 3 in Q_3 starting at $(0, 0, 0)$, there is always a unique way to complete it to a Hamiltonian cycle. It might be helpful to use automorphisms of Q_3 to reduce the number of cases considered. You may use, without proof, that if $\sigma\colon [3] \to [3]$ is a bijection, then the function $(x_1, x_2, x_3) \mapsto (x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$ is an automorphism of Q_3.]