Math 475, Fall 2015
Homework 2
Due: Friday, Sep. 18

Notation: $[n] = \{1, 2, \ldots, n\}$.

(1) Bóna 3.26: How many ways are there to list the letters of the word ALABAMA?

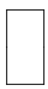
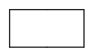
(2) Bóna 3.30: How many four-digit positive integers are there in which all digits are different?

(3) Bóna 3.41: We want to select three subsets A, B, and C of $[n]$ so that $A \subseteq C$, $B \subseteq C$, and $A \cap B \neq \emptyset$. In how many different ways can we do this?

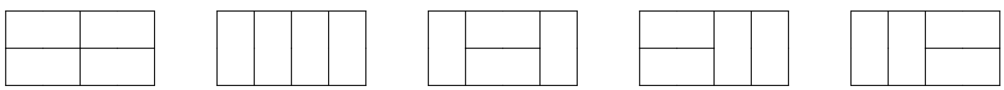
(4) Fix a positive integer $n \geq 1$. Let A_1 be the set of subsets $S \subseteq [n]$ with no consecutive elements, i.e., if $i \in S$, then $i + 1 \notin S$.

For example, when $n = 3$, $|A_1| = 5$ and A_1 is the following set of subsets:

$$\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}.$$

Let A_2 be the set of ways of tiling the $2 \times (n + 1)$ rectangle with the shapes: 2×1 rectangle  and 1×2 rectangle  without any overlaps.

For example, when $n = 3$, $|A_2| = 5$ and A_2 is the following set of tilings:



Find a bijection between A_1 and A_2.

(5) Let n and k be positive integers. Show that the number of ordered collections $(X_1, \ldots, X_k)$, where each X_i is a subset of $[n]$, and $X_1 \cap X_2 \cap \cdots \cap X_k = \emptyset$ (i.e., there is no element which is in all of the X_i) is $(2^k - 1)^n$.

For example, when $k = 2$ and $n = 2$, here are the 9 ordered collections:

$$\begin{array}{rrr} (\emptyset, \emptyset) & (\emptyset, \{1\}) & (\emptyset, \{2\}) \\ (\emptyset, \{1, 2\}) & (\{1\}, \emptyset) & (\{2\}, \emptyset) \\ (\{1, 2\}, \emptyset) & (\{1\}, \{2\}) & (\{2\}, \{1\}). \end{array}$$

(6) How many 6-card hands from a standard deck of cards (i.e., 4 suits and 13 face values) contain exactly 2 pairs? (In other words, there are 2 cards with the same face value, another 2 cards with the same face value, but these two face values are different, and the remaining 2 cards have different face values from these two pairs and each other.)